INEQUALITIES RELATED TO FREE ENTROPY DERIVED FROM RANDOM MATRIX APPROXIMATION

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ABSTRACT. Biane proved the free analog of the logarithmic Sobolev inequality for probability measures on $\mathbf{R}$ by means of random matrix approximation procedure. We show that the same method can be applied to reprove Biane and Voiculescu's free analog of Talagrand's transportation cost inequality for measures on $\mathbf{R}$. Furthermore, we prove the free analogs of the logarithmic Sobolev inequality and the transportation cost inequality for measures on $\mathbf{T}$ as well by extending the method to special unitary random matrices.

INTRODUCTION

Since its first systematic study done by L. Gross [12] in 1975, the logarithmic Sobolev inequality (LSI) has been discussed by many authors in various contexts, in particular, in close connection with the notions of hypercontractivity and spectral gap. An LSI can be understood to compare the relative Fisher information with the relative entropy. Among other things, we here refer to the LSI due to D. Bakry and M. Emery [1] in the general Riemannian manifold setting, which is of quite use for our present purpose. Another interesting inequality was presented by M. Talagrand [28] in 1996, called the transportation cost inequality (TCI). A TCI compares the (quadratic) Wasserstein distance $W(\mu,\nu)$ between probability measures μ,ν (for the definition see (4.1) in §4 of this paper) with $\sqrt{S(\mu,\nu)}$, the square root of the relative entropy. Indeed, in [28] Talagrand proved the inequality $W(\mu,\nu) \le \sqrt{S(\mu,\nu)}$ when ν is the standard Gaussian measure on $\mathbf{R}^n$, and an exposition in the case of more general ν can be found in [21] for example. On the other hand, in [25] F. Otto and C. Villani succeeded in discovering links between the LSI and the TCI in the Riemannian manifold setting. This, combined with [1], implies the TCI in the same situation as Bakry and Emery's LSI. See [20, 21, 29] for more about these classical LSI and TCI as well as related topics.

The relative free entropy $\widetilde{\Sigma}_Q(\mu)$ and the relative free Fisher information $\Phi_Q(\mu)$ were introduced by Ph. Biane and R. Speicher [5] for $\mu \in \mathcal{M}(\mathbf{R})$, the probability measures on $\mathbf{R}$, relative to a real continuous function Q on $\mathbf{R}$, where Q has a certain growth in the case of $\widetilde{\Sigma}_Q(\mu)$ and it is a C^1 function in the case of $\Phi_Q(\mu)$. Note that $\widetilde{\Sigma}_Q(\mu)$ is regarded as the relative version of the free entropy $\Sigma(\mu)$ introduced by D. Voiculescu [30] as the classical relative entropy is the relative version of the Boltzmann-Gibbs entropy, while $\Phi_Q(\mu)$ in the case $Q \equiv 0$ reduces to the free Fisher information $\Phi(\mu)$ in [30]. (The "free relative entropy" $\Sigma(\mu,\nu)$ for two measures was introduced in [13] from a slightly different viewpoint.) In this paper we introduce the relative free entropy $\widetilde{\Sigma}_Q(\mu)$ and the relative free Fisher information $F_Q(\mu)$ for $\mu \in \mathcal{M}(\mathbf{T})$, the probability measures on the unit circle $\mathbf{T}$, as well relative to a real continuous function Q on $\mathbf{T}$ (being a C^1 function for $F_Q(\mu)$). When $Q \equiv 0$ the quantity $F_Q(\mu)$ becomes

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the free Fisher information $F(\mu)$ introduced by Voiculescu [33]. An important fact is that the relative free entropy $\widetilde{\Sigma}_Q(\mu)$ is the rate function (or the so-called weighted logarithmic integral up to an additive constant) of a large deviation for the empirical eigenvalue distribution of a certain random matrix. Indeed, $\widetilde{\Sigma}_Q(\mu)$ for $\mu \in \mathcal{M}(\mathbf{R})$ is the good rate function of large deviation principle for the $n \times n$ selfadjoint random matrix determined by the function Q, while $\widetilde{\Sigma}_Q(\mu)$ for $\mu \in \mathcal{M}(\mathbf{T})$ is that for the $n \times n$ (special) unitary random matrix associated with Q. The definitions of these quantities as well as related matters are collected in the first §1 of this paper.

Voiculescu's inequality in [32, Proposition 7.9] is the first free probabilistic analog of the LSI. Extending its single variable case (see (2.4) in §2), Biane obtained in [4] the following free LSI:

$$\widetilde{\Sigma}_Q(\mu) \le \frac{1}{2\rho}\Phi_Q(\mu) \quad \text{for } \mu \in \mathcal{M}(\mathbf{R})$$

if $Q''(x) \ge \rho$ on $\mathbf{R}$ with a constant $\rho > 0$. To prove this, Biane applied the classical LSI on the Euclidean space to the related selfadjoint random matrices as mentioned above and used the weak convergence of their mean eigenvalue distributions. Although the differentiability assumption of Q is not quite explicitly written in [4], Biane's free LSI is certainly valid if Q is a C^1 function such that $Q(x) - \frac{\rho}{2}x^2$ is convex on $\mathbf{R}$. For the sake of completeness, in §2 we give a proof of this general case by a usual approximation technique.

The first main aim of this paper is to show the variant of Biane's free LSI for measures on $\mathbf{T}$. In §3 we prove

$$\widetilde{\Sigma}_Q(\mu) \le \frac{1}{1+2\rho}F_Q(\mu) \quad \text{for } \mu \in \mathcal{M}(\mathbf{T})$$

if Q is a C^1 function on $\mathbf{T}$ such that $Q\left(e^{\sqrt{-1}t}\right) - \frac{\rho}{2}t^2$ is convex on $\mathbf{R}$ with a constant $\rho > -1/2$. The proof is based on random matrix approximation. We can apply Bakry and Emery's classical LSI on the special unitary group $\mathrm{SU}(n)$, a Riemannian manifold, to the related $n \times n$ special unitary random matrices and pass to the scaling limit as n goes to ∞. Here, we need the convergence of the empirical eigenvalue distribution of the random matrix not only in the mean but also in the almost sure sense that is a consequence of the corresponding large deviation principle. Although the large deviation theorem (Theorem 1.2 below) for "special" unitary random matrices is essentially same as that for unitary random matrices shown in [16], the proof is a bit more complicated so that we sketch it in Appendix for the convenience of the reader. We also need a few stuffs from differential geometry, in particular, the exact computation of the Ricci curvature tensor of $\mathrm{SU}(n)$ (with respect to the Riemannian structure associated with the usual trace on $M_n(\mathbf{C})$) to check the so-called Bakry and Emery criterion (see §§1.7).

In [6] Biane and Voiculescu obtained the free analog of Talagrand's TCI for compactly supported $\mu \in \mathcal{M}(\mathbf{R})$ as follows:

$$W(\mu, \gamma_{0,2}) \le \sqrt{-\Sigma(\mu) + \int \frac{x^2}{2}\,d\mu(x) - \frac{3}{4}},$$

where $\gamma_{0,2}$ denotes the standard semicircular distribution (with radius 2). Their proof involves the free process and the complex Burgers' equation, and it is a realization of free probability parallel of not only the result itself but also the proof in [25]. The proof itself justifies the above inequality to be the right free analog of Talagrand's TCI.

Our second main aim is to reprove Biane and Voiculescu's free TCI in a slightly more general setting by making use of random matrix approximation and furthermore to give a free TCI for measures on $\mathbf{T}$ in a similar way. This aim is our initial motivation; we first wanted

to find another proof to Biane and Voiculescu's TCI by use of random matrix approximation on the lines of so-called Voiculescu's heuristics in [30] and to justify Biane and Voiculescu's TCI as the right free analog from the viewpoint of random matrix theory. In §4 we prove the free TCI

$$W(\mu,\mu_Q)\le\sqrt{\frac{1}{\rho}\widetilde{\Sigma}_Q(\mu)}\quad\text{for compactly supported } \mu\in\mathcal{M}(\mathbf{R})$$

if Q is a real function on $\mathbf{R}$ such that $Q(x)-\frac{\rho}{2}x^2$ is convex with a constant $\rho>0$ and μ_Q is the equilibrium measure associated with Q (or the unique minimizer of $\widetilde{\Sigma}_Q(\mu)$ for $\mu\in\mathcal{M}(\mathbf{R})$). When $Q(x)=x^2/2$ and $\rho=1$, this becomes Biane and Voiculescu's TCI. To prove this, we first suppose that μ is supported in $[-R,R]$ and that $Q_\mu(x):=2\int_{\mathbf{R}}\log|x-y|\,d\mu(y)$ is continuous on $\mathbf{R}$. We consider two $n\times n$ selfadjoint random matrices; one is associated with Q, and the other is associated with Q_μ and restricted on the $n\times n$ selfadjoint matrices with the operator norm $\le R$. Then, these random matrices are probability measures on the space of $n\times n$ selfadjoint matrices ($\cong\mathbf{R}^{n^2}$), and the classical TCI for these measures asymptotically approaches, as n goes to ∞, to the free TCI we want. The case of general compactly supported $\mu\in\mathcal{M}(\mathbf{R})$ can be treated by an approximation technique. Furthermore, as presented in §5, a similar method using special unitary random matrices can work to prove the free TCI

$$W(\mu,\mu_Q)\le\sqrt{\frac{2}{1+2\rho}\widetilde{\Sigma}_Q(\mu)}\quad\text{for } \mu\in\mathcal{M}(\mathbf{T})$$

if Q is such a real function on $\mathbf{T}$ as in the free LSI, that is, $Q\left(e^{\sqrt{-1}t}\right)-\frac{\rho}{2}t^2$ is convex on $\mathbf{R}$ with $\rho>-1/2$. Here, $W(\mu,\mu_Q)$ is the Wasserstein distance with respect to the geodesic distance (or the angular distance) on $\mathbf{T}$. In the particular case where $Q\equiv 0$ and $\rho=0$, we have $W(\mu,d\theta/2\pi)\le\sqrt{2\Sigma(\mu)}$.

In this way, we clarify the advantage of random matrix approximation procedure in studying free probabilistic analogs of certain classical theories involving relative entropy and/or Fisher information. The present paper may be regarded as one more attempt subsequent to [2, 13] toward rigorous realizations of Voiculescu's heuristics in [30] which claims that the classical entropy of random matrices, if suitably arranged, asymptotically converges to the free entropy of the limit distribution as the matrix size goes to infinity.

The final §6 is a collection of remarks, examples and related results; in particular, we give the variants of the above free LSI and TCI for measures on the half line $\mathbf{R}^+$.

1. Preliminaries

The purpose of this preliminary section is to summarize, for the convenience of the reader, the basic notions and the results which will be needed later. We will use them with no explicit explanation in the main part of this paper.

1.1. **Notations.** The set of all Borel probability measures on a Polish space $\mathcal{X}$ is denoted by $\mathcal{M}(\mathcal{X})$. The Dirac measure at a point $x\in\mathcal{X}$ is denoted by δ_x as usual. For $\mu,\nu\in\mathcal{M}(\mathcal{X})$, the *relative entropy* of μ with respect to ν is denoted by $S(\mu,\nu)$, which is defined by

$$S(\mu,\nu):=\int_{\mathcal{X}}\log\frac{d\mu}{d\nu}\,d\mu=\int_{\mathcal{X}}\frac{d\mu}{d\nu}\log\frac{d\mu}{d\nu}\,d\nu \tag{1.1}$$

when μ is absolutely continuous with respect to ν; otherwise $S(\mu,\nu):=+\infty$.

The usual trace on $M_n(\mathbf{C})$, the $n\times n$ complex matrices, is denoted by Tr_n. The Hilbert-Schmidt norm on $M_n(\mathbf{C})$ induced from Tr_n is denoted by $\|\cdot\|_{HS}$, i.e., $\|A\|_{HS}:=\mathrm{Tr}_n(A^*A)^{1/2}$ for $A\in M_n(\mathbf{C})$. Let M_n^{sa} denote the set of all $n\times n$ self-adjoint matrices, $\mathrm{U}(n)$ the group

of all $n \times n$ unitaries, and $\mathrm{SU}(n)$ the special unitary group of order n, i.e., the group of all $n \times n$ unitaries whose determinants are equal to one.

1.2. **Free entropy and free Fisher information for measures on R.** The notions of free entropy and free Fisher information are the free probabilistic analogs of the Boltzmann-Gibbs entropy and the Fisher information in classical theory. For each $\mu \in \mathcal{M}(\mathbf{R})$, Voiculescu [30] introduced the *free entropy* of μ

$$\Sigma(\mu) := \iint_{\mathbf{R}^2} \log|x-y|\, d\mu(x)\, d\mu(y),$$

which is the minus of the so-called *logarithmic energy* of μ useful in potential theory (see [26]). It is the "main component" of the free entropy $\chi(\mu)$ of μ introduced in [31]:

$$\chi(\mu) = \Sigma(\mu) + \frac{3}{4} + \frac{1}{2}\log 2\pi. \tag{1.2}$$

Assume that $\mu \in \mathcal{M}(\mathbf{R})$ has the density $p = d\mu/dx$ (with respect to the Lebesgue measure dx) belonging to the L^3-space $L^3(\mathbf{R}) := L^3(\mathbf{R}, dx)$. In [30] Voiculescu also introduced the *free Fisher information* of μ

$$\Phi(\mu) := \frac{4\pi^2}{3}\int_{\mathbf{R}} p(x)^3\, dx = \frac{4\pi^2}{3}\|p\|_3^3.$$

The *Hilbert transform* of p

$$(Hp)(x) := \lim_{\varepsilon \searrow 0} \int_{|x-t|>\varepsilon} \frac{p(t)}{x-t}\, dt \tag{1.3}$$

plays an important role in the study of free Fisher information. The limit in (1.3) really exists for a.e. $x \in \mathbf{R}$ (as long as $p \in L^q(\mathbf{R})$ with $1 < q < \infty$), and it is known that $p \in L^q(\mathbf{R})$ implies $Hp \in L^q(\mathbf{R})$ for each $1 < q < \infty$. See [19, Chapter VI] for the Hilbert transform on $\mathbf{R}$. As shown in [30, Lemma 3.3] we see that

$$\int_{\mathbf{R}} ((Hp)(x))^2\, p(x)\, dx = \frac{\pi^2}{3}\int_{\mathbf{R}} p(x)^3\, dx, \tag{1.4}$$

and hence the free Fisher information has an alternative description:

$$\Phi(\mu) = 4\int_{\mathbf{R}} ((Hp)(x))^2\, p(x)\, dx = 4\int_{\mathbf{R}} ((Hp)(x))^2\, d\mu(x).$$

Here, we should remark that the Hilbert transform is usually defined with an additional multiple constant $1/\pi$ and $\int_{\mathbf{R}} ((Hp)(x))^2\, p(x) dx = \frac{1}{3}\int_{\mathbf{R}} p(x)^3 dx$ holds instead of (1.4) in this case.

Let Q be a real-valued C^1 function on $\mathbf{R}$. For each $\mu \in \mathcal{M}(\mathbf{R})$, Biane and Speicher [5, §6] introduced the *relative free Fisher information* $\Phi_Q(\mu)$ of μ relative to Q, and it is defined to be

$$\Phi_Q(\mu) := 4\int_{\mathbf{R}} \left((Hp)(x) - \frac{1}{2}Q'(x)\right)^2 d\mu(x) \tag{1.5}$$

when μ has the density $p = d\mu/dx$ belonging to $L^3(\mathbf{R})$; otherwise to be $+\infty$.

1.3. **Free entropy and free Fisher information for measures on T.** For each $\mu \in \mathcal{M}(\mathbf{T})$, the *free entropy* $\Sigma(\mu)$ of μ is defined in the same manner as in the real line case; that is,

$$\Sigma(\mu) := \iint_{\mathbf{T}^2} \log|\zeta - \eta| \, d\mu(\zeta) \, d\mu(\eta)$$

([33, §§10.7], [15]). For its justification to be a right quantity, see [33, Proposition 10.8] in relation to the free Fisher information as well as [15, Proposition 1.4], [16] from the microstate approach or large deviation principle.

Assume that $\mu \in \mathcal{M}(\mathbf{T})$ has the density $p = d\mu/d\zeta$ with respect to the Haar probability measure $d\zeta = d\theta/2\pi$, $\zeta = e^{\sqrt{-1}\theta}$ with $\theta \in [-\pi, \pi)$ and further that p belongs to the L^3-space $L^3(\mathbf{T}) := L^3(\mathbf{T}, d\zeta)$. As in the real line case, the Hilbert transform of p

$$(Hp)\left(e^{\sqrt{-1}\theta}\right) := \lim_{\varepsilon \searrow 0} \int_{\varepsilon \le |t| < \pi} \frac{p\left(e^{\sqrt{-1}(\theta - t)}\right)}{\tan\left(\frac{t}{2}\right)} \frac{dt}{2\pi} \tag{1.6}$$

is important. The principle value limit in (1.6) exists for a.e. (as long as $p \in L^1(\mathbf{T})$), and it is known that $p \in L^q(\mathbf{T})$ implies $Hp \in L^q(\mathbf{T})$ as well for each $1 < q < \infty$. See [19, Chapter V] for detailed accounts on the Hilbert transform on $\mathbf{T}$. Following Voiculescu [33, §§8.9] we call the quantity

$$F(\mu) := \int_{\mathbf{T}} ((Hp)(\zeta))^2 \, d\mu(\zeta) = \int_{\mathbf{T}} ((Hp)(\zeta))^2 \, p(\zeta) \, d\zeta$$

the *free Fisher information* of μ. When μ has no such density as above, $F(\mu)$ is defined to be $+\infty$. By [33, Corollary 8.8 and Definition 8.9] the free Fisher information can be written as

$$F(\mu) = \frac{1}{3}\left(-1 + \int_{\mathbf{T}} p(\zeta)^3 \, d\zeta\right).$$

Let Q be a real-valued C^1 function on $\mathbf{T}$. As in the case of measures on $\mathbf{R}$, for each $\mu \in \mathcal{M}(\mathbf{T})$ we define the *relative free Fisher information* $F_Q(\mu)$ to be

$$F_Q(\mu) := \int_{\mathbf{T}} \left((Hp)(\zeta) - Q'(\zeta)\right)^2 d\mu(\zeta) - \left(\int_{\mathbf{T}} Q'(\zeta) \, d\mu(\zeta)\right)^2 \tag{1.7}$$

when μ has the density $p = d\mu/d\zeta$ belonging to $L^3(\mathbf{T})$; otherwise to be $+\infty$. Here, Q' means the derivative of $Q(e^{\sqrt{-1}\theta})$ in θ, i.e., $Q'(e^{i\theta}) = \frac{d}{d\theta} Q(e^{\sqrt{-1}\theta})$. Slight difference between the two formulas (1.5) and (1.7) is worth notice.

1.4. **Large deviations for self-adjoint random matrices.** Let Q be a real-valued continuous function on $\mathbf{R}$ such that

$$\lim_{|x| \to +\infty} |x| \exp(-\varepsilon Q(x)) = 0 \quad \text{for every } \varepsilon > 0. \tag{1.8}$$

The *weighted energy integral* associated with Q is defined by

$$E_Q(\mu) := -\Sigma(\mu) + \int_{\mathbf{R}} Q(x) \, d\mu(x) \quad \text{for } \mu \in \mathcal{M}(\mathbf{R}).$$

According to a fundamental result in the theory of weighted potentials (see [26, I.1.3]), there exists a unique $\mu_Q \in \mathcal{M}(\mathbf{R})$ such that

$$E_Q(\mu_Q) = \inf\left\{E_Q(\mu) : \mu \in \mathcal{M}(\mathbf{R})\right\},$$

and $E_Q(\mu_Q)$ is finite (hence so is $\Sigma(\mu_Q)$). Moreover, μ_Q is known to be compactly supported. The minimizer μ_Q is sometimes called the *equilibrium measure* associated with Q. Set $B(Q) := -E_Q(\mu_Q)$ so that the function

$$-\Sigma(\mu) + \int_{\mathbf{R}} Q(x)\,d\mu(x) + B(Q) \quad \text{for } \mu \in \mathcal{M}(\mathbf{R}) \tag{1.9}$$

is non-negative and is zero only when $\mu = \mu_Q$. It is well known that if $Q(x) = 2x^2/r^2$ with $r > 0$, then the equilibrium measure (or the unique minimizer) μ_Q is the $(0, r^2/4)$-*semicircular distribution* $\gamma_{0,r}$ (with variance $r^2/4$):

$$d\gamma_{0,r}(x) := \frac{2}{\pi r^2}\sqrt{r^2 - x^2}\,\chi_{[-r,r]}(x)\,dx. \tag{1.10}$$

For each $n \in \mathbf{N}$ define $\lambda_n(Q) \in \mathcal{M}(M_n^{sa})$, the $n \times n$ *self-adjoint random matrix* associated with Q, by

$$d\lambda_n(Q)(A) := \frac{1}{Z_n(Q)} \exp\bigl(-n\mathrm{Tr}_n(Q(A))\bigr)\,dA,$$

where dA means the "Lebesgue measure" on $M_n^{sa} \cong \mathbf{R}^{n^2}$, i.e.,

$$dA := \prod_{i=1}^n dA_{ii} \prod_{i<j} d\,(\mathrm{Re}\,A_{ij})\,d\,(\mathrm{Im}\,A_{ij}) \quad \text{with } A = [A_{ij}],$$

$Q(A)$ is the usual functional calculus and $Z_n(Q)$ is a normalization constant. It is known (see [22, 17] for example) that the *joint eigenvalue distribution* on $\mathbf{R}^n$ of $\lambda_n(Q)$ is given as

$$d\tilde{\lambda}_n(Q)(x_1, \dots, x_n) := \frac{1}{\widetilde{Z}_n(Q)} \exp\left(-n \sum_{i=1}^n Q(x_i)\right) \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^n dx_i$$

with a new normalization constant $\widetilde{Z}_n(Q)$. Moreover, the *mean eigenvalue distribution* on $\mathbf{R}$ of $\lambda_n(Q)$ is defined by

$$\hat{\lambda}_n(Q) := \int \cdots \int_{\mathbf{R}^n} \frac{1}{n}\,(\delta_{x_1} + \cdots + \delta_{x_n})\,d\tilde{\lambda}_n(Q)(x_1, \dots, x_n).$$

In [2] Ben Arous and Guionnet showed the large deviation principle for the empirical eigenvalue distribution of the standard self-adjoint Gaussian random matrix (i.e., $\lambda_n(Q)$ with $Q(x) = x^2/2$). The following is its slight generalization given in [17, 5.4.3]: When $(x_1, \dots, x_n)$ is distributed according to $\tilde{\lambda}_n(Q)$, the *empirical eigenvalue distribution*

$$\frac{1}{n}\,(\delta_{x_1} + \cdots + \delta_{x_n}) \tag{1.11}$$

satisfies the large deviation principle in the scale $1/n^2$ and the good rate function is given by (1.9). Furthermore, one has $B(Q) = \lim_{n\to\infty} \frac{1}{n^2} \log \widetilde{Z}_n(Q)$, i.e.,

$$B(Q) = \lim_{n\to\infty} \frac{1}{n^2} \log \int \cdots \int_{\mathbf{R}^n} \exp\left(-n \sum_{i=1}^n Q\,(x_i)\right) \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^n dx_i. \tag{1.12}$$

See [8, 9] for general theory of large deviations. Since μ_Q is the unique minimizer of (1.9), the random measure (1.11) converges in the weak topology to μ_Q almost surely, and hence

$$\hat{\lambda}_n(Q) \longrightarrow \mu_Q \quad \text{weakly;} \tag{1.13}$$

see [17, p. 211] and also [7]. From the viewpoint of the large deviation theory of level-2 (see [8, 9]), the function (1.9) can be regarded as a kind of free analog of the relative entropy with

respect to its unique minimizer μ_Q. Thus, following Biane and Speicher [5, §6] and Biane [4, §3], we call the function (1.9) the *relative free entropy* (or *modified free entropy*) of μ relative to Q, which is denoted by $\widetilde{\Sigma}_Q(\mu)$; that is,

$$\widetilde{\Sigma}_Q(\mu) := -\Sigma(\mu) + \int_{\mathbf{R}} Q(x)\,d\mu(x) + B(Q) \quad \text{for } \mu \in \mathcal{M}(\mathbf{R}). \tag{1.14}$$

We do *not* call this the "free relative entropy" introduced in [13], a slightly different relative entropy-like quantity $\Sigma(\mu,\nu)$ for two probability measures in the framework of free probability. Indeed, the free relative entropy $\Sigma(\mu,\nu)$ for $\mu,\nu \in \mathcal{M}(\mathbf{R})$ is defined as

$$\Sigma(\mu,\nu) := \iint_{\mathbf{R}^2} \log|x-y|\,d(\mu-\nu)(x)\,d(\mu-\nu)(y).$$

But it is known (see [13, (2.7)]) that

$$\Sigma(\mu,\mu_Q) = \widetilde{\Sigma}_Q(\mu)$$

if the support of μ is included in that of μ_Q.

1.5. **Large deviations for restricted self-adjoint random matrices.** In the course of finding a right free analog of relative entropy, another random matrix model associated with Q and $R > 0$ was introduced in [13]. Here, Q is an arbitrary real-valued continuous function whose domain includes $[-R,R]$. The self-adjoint random matrix $\lambda_n(Q;R) \in \mathcal{M}(M_n^{sa})$ restricted on a compact subset $\{A \in M_n^{sa} : \|A\|_\infty \le R\}$ is defined by

$$d\lambda_n(Q;R)(A) := \frac{1}{Z_n(Q;R)} \exp\bigl(-n\mathrm{Tr}_n(Q(A))\bigr)\chi_{\{\|A\|_\infty \le R\}}(A)\,dA$$

with a normalization constant $Z_n(Q;R)$. In the above, $\|\cdot\|_\infty$ means the operator norm. The joint eigenvalue distribution supported in $[-R,R]^n$ of $\lambda_n(Q;R)$ is given as

$$\begin{aligned} &d\tilde{\lambda}_n(Q;R)(x_1,\dots,x_n) \\ &\quad := \frac{1}{\widetilde{Z}_n(Q;R)} \exp\left(-n\sum_{i=1}^n Q(x_i)\right) \prod_{i<j}(x_i-x_j)^2 \prod_{i=1}^n \chi_{[-R,R]}(x_i)\,dx_i \end{aligned}$$

with a new normalization constant $\tilde{Z}_n(Q;R)$. Its mean eigenvalue distribution $\hat{\lambda}_n(Q;R)$ supported in $[-R,R]$ is defined as in §§1.4. As in the case of $\lambda_n(Q)$, the following large deviation theorem holds: The finite limit $B(Q;R) := \lim_{n\to\infty} \frac{1}{n^2}\log \tilde{Z}_n(Q;R)$ exists, and when $(x_1,\dots,x_n)$ is distributed according to $\tilde{\lambda}_n(Q;R)$, the empirical eigenvalue distribution (1.11) satisfies the large deviation principle in the scale $1/n^2$ with the rate function

$$-\Sigma(\mu) + \int_{[-R,R]} Q(x)\,d\mu(x) + B(Q;R) \quad \text{for } \mu \in \mathcal{M}([-R,R]). \tag{1.15}$$

The proof of this large deviation principle is similar to [17, 5.4.3 and 5.5.1] as noticed in [13]. In this setting, there also exists a unique minimizer $\mu_{Q,R} \in \mathcal{M}([-R,R])$ of the rate function (1.15), whose value at $\mu_{Q,R}$ is zero. If $R > 0$ is chosen so that μ_Q in §§1.4 is supported in $[-R,R]$, then $\mu_Q = \mu_{Q,R}$ is seen by comparing the two rate functions, and hence $B(Q) = B(Q;R)$; this assertion is essentially same as in [31, Proposition 2.4] in the single variable case.

1.6. **Large deviations for special unitary random matrices.** Let Q be a real-valued continuous function on $\mathbf{T}$. Similarly to the real line case in §§1.4, the weighted energy integral

$$-\Sigma(\mu) + \int_{\mathbf{T}} Q(\zeta)\, d\mu(\zeta) \quad \text{for } \mu \in \mathcal{M}(\mathbf{T})$$

admits a unique minimizer $\mu_Q \in \mathcal{M}(\mathbf{T})$ (or the equilibrium measure associated with Q). Set $B(Q) := \Sigma(\mu_Q) - \int_{\mathbf{T}} Q(\zeta)\, d\mu_Q(\zeta)$. It is known ([16]) that the function

$$-\Sigma(\mu) + \int_{\mathbf{T}} Q(\zeta)\, d\mu(\zeta) + B(Q) \quad \text{for } \mu \in \mathcal{M}(\mathbf{T})$$

is the rate function of the large deviation for the empirical eigenvalue distribution of an $n \times n$ *unitary random matrix*

$$d\lambda_n^{\mathrm{U}}(Q)(U) := \frac{1}{Z_n^{\mathrm{U}}(Q)} \exp\Bigl(-n\mathrm{Tr}_n(Q(U))\Bigr)\, dU,$$

where dU is the Haar probability measure on $\mathrm{U}(n)$, $Q(U)$ is defined via functional calculus and $Z_n^{\mathrm{U}}(Q)$ is a normalization constant. Furthermore,

$$B(Q) = \lim_{n\to\infty} \frac{1}{n^2} \log \int \cdots \int_{\mathbf{T}^n} \exp\left(-n\sum_{i=1}^n Q(\zeta_i)\right) \prod_{1\le i<j\le n} |\zeta_i - \zeta_j|^2 \prod_{i=1}^n d\zeta_i$$

where $d\zeta_i = d\theta_i/2\pi$ for $\zeta_i = e^{\sqrt{-1}\theta_i}$. However, the above unitary random matrix $\lambda_n^{\mathrm{U}}(Q)$ is not suitable for our present purpose as will be explained in §§1.7. Thus, we need to modify the above large deviation to the setup of $\mathrm{SU}(n)$.

Now, we begin with the joint eigenvalue distribution of the Haar probability measure on the special unitary group $\mathrm{SU}(n)$. Note that the n eigenvalues $\zeta_1, \dots, \zeta_n$ of $U \in \mathrm{SU}(n)$ satisfy $\zeta_1 \cdots \zeta_n = 1$, i.e., $\zeta_n = (\zeta_1 \cdots \zeta_{n-1})^{-1}$ so that the joint density must be a permutation-invariant distribution of $(\zeta_1, \dots, \zeta_{n-1}) \in \mathbf{T}^{n-1}$. The following explicit form of the density seems a folklore for specialists, and in fact, it is easily derived from the Weyl integration formula familiar in representation theory; see [18, p. 104] for example.

Lemma 1.1. *The joint eigenvalue distribution on $\mathbf{T}^{n-1}$ of the Haar probability measure on $\mathrm{SU}(n)$ is*

$$\frac{1}{n!} \prod_{1\le i<j\le n} |\zeta_i - \zeta_j|^2 \prod_{i=1}^{n-1} d\zeta_i \quad \text{with } \zeta_n = (\zeta_1 \cdots \zeta_{n-1})^{-1},$$

or

$$\frac{1}{n!(2\pi)^{n-1}} \prod_{1\le i<j\le n} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^2 \prod_{i=1}^{n-1} d\theta_i$$
$$\text{with } \theta_n = -(\theta_1 + \cdots + \theta_{n-1}) \pmod{2\pi}.$$

Let Q be a real-valued continuous function on $\mathbf{T}$. For each $n \in \mathbf{N}$ define $\lambda_n(Q) \in \mathcal{M}(\mathrm{SU}(n))$, the $n \times n$ *special unitary random matrix* associated with Q, by

$$d\lambda_n^{\mathrm{SU}}(Q)(U) := \frac{1}{Z_n^{\mathrm{SU}}(Q)} \exp\bigl(-n\mathrm{Tr}_n(Q(U))\bigr)\, dU, \tag{1.16}$$

where dU is the Haar probability measure on $\mathrm{SU}(n)$ and $Z_n^{\mathrm{SU}}(Q)$ is a normalization constant. By Lemma 1.1 the joint eigenvalue distribution on $\mathbf{T}^{n-1}$ of $\lambda_n^{\mathrm{SU}}(Q)$ is given as

$$d\tilde{\lambda}_n^{\mathrm{SU}}(Q)(\zeta_1,\dots,\zeta_{n-1}) = \frac{1}{\widetilde{Z}_n^{\mathrm{SU}}(Q)} \exp\left(-n\sum_{i=1}^n Q(\zeta_i)\right) \prod_{1\le i<j\le n} |\zeta_i-\zeta_j|^2 \prod_{i=1}^n d\zeta_i$$
$$\text{with} \quad \zeta_n = (\zeta_1\cdots\zeta_{n-1})^{-1}.$$

The next theorem is the large deviation principle for the empirical eigenvalue distribution of $\lambda_n^{\mathrm{SU}}(Q)$, whose proof, based on the explicit form of the density of $\tilde{\lambda}_n^{\mathrm{SU}}(Q)$, will be sketched in Appendix for the convenience of the reader.

Theorem 1.2. *The finite limit $B(Q) := \lim_{n\to\infty} \frac{1}{n^2}\log \widetilde{Z}_n^{\mathrm{SU}}(Q)$ exists. When $(\zeta_1,\dots,\zeta_{n-1})$ is distributed on $\mathbf{T}^{n-1}$ according to $\tilde{\lambda}_n^{\mathrm{SU}}(Q)$, the empirical distribution $\frac{1}{n}(\delta_{\zeta_1}+\cdots+\delta_{\zeta_{n-1}}+\delta_{\zeta_n})$ with $\zeta_n = (\zeta_1\cdots\zeta_{n-1})^{-1}$ satisfies the large deviation principle in the scale $1/n^2$ with the rate function*

$$\widetilde{\Sigma}_Q(\mu) := -\Sigma(\mu) + \int_{\mathbf{T}} Q(\zeta)\,d\mu(\zeta) + B(Q) \quad \text{for } \mu\in\mathcal{M}(\mathbf{T}). \tag{1.17}$$

Furthermore, there exists a unique minimizer $\mu_Q \in \mathcal{M}(\mathbf{T})$ of the rate function so that $\widetilde{\Sigma}_Q(\mu_Q) = 0$.

As before, we call the rate function (1.17) the *relative free entropy* of μ with respect to Q, which is denoted by $\widetilde{\Sigma}_Q(\mu)$ as in (1.14).

1.7. **Ricci curvature tensor of** $\mathrm{SU}(n)$**.** Let M be a smooth complete Riemannian manifold of dimension m, and let $\mathrm{Ric}(M)$ denote the *Ricci curvature tensor* of M. For a real-valued C^2 function Ψ on M, the *Hessian* of Ψ is denoted by $\mathrm{Hess}(\Psi)$. Our arguments in §3 and §5 will need to verify the so-called *Bakry and Emery criterion* with a positive constant ρ:

$$\mathrm{Ric}(M) + \mathrm{Hess}(\Psi) \ge \rho I_m; \tag{1.18}$$

see [1] and Theorem 2.1 below.

The Ricci curvature tensor of $\mathrm{U}(n)$ is known to be degenerate, while that of $\mathrm{SU}(n)$ to be of positive constant (see [23], a nice reference for the topic) and a straightforward computation shows that the Ricci curvature tensor of $\mathrm{SU}(n)$ with respect to the Riemannian structure associated with Tr_n is

$$\mathrm{Ric}\,(\mathrm{SU}(n)) = \frac{n}{2} I_{n^2-1}. \tag{1.19}$$

This is the reason why we have presented Theorem 1.2 with use of $\mathrm{SU}(n)$ instead of $\mathrm{U}(n)$.

1.8. **Differentiability of trace functions.** A derivative formula as well as the higher order differentiability for a certain kind of trace functions will be essential in proving the main result (Theorem 3.3) in §3. The topic seems rather familiar to specialists, however we can find no appropriate literature. Here, a lemma is recorded in a form tailor-made for our later use without full generality.

Let $f(t)$ be a real-valued function on an interval (a,b), and let $\lambda_1,\lambda_2,\dots$ be distinct points in (a,b). The *divided differences* $f^{[r]}$ for $r=0,1,2,\dots$ are recursively introduced as follows: $f^{[0]}(\lambda_1) := f(\lambda_1)$ and

$$f^{[r]}(\lambda_1,\lambda_2,\dots,\lambda_{r+1}) := \frac{f^{[r-1]}(\lambda_1,\lambda_2,\dots,\lambda_r) - f^{[r-1]}(\lambda_2,\dots,\lambda_r,\lambda_{r+1})}{\lambda_1-\lambda_{r+1}}.$$

When λ_i's are not necessarily distinct, $f^{[r]}(\lambda_1, \lambda_2, \dots, \lambda_{r+1})$ can be defined by continuity as long as $f \in C^r(a,b)$; for example, $f^{[2]}(\lambda, \lambda) = f'(\lambda)$ and $f^{[3]}(\lambda, \lambda, \lambda) = f''(\lambda)/2$. See [10, §II.2] for basic properties of divided differences. Let $A \in M_n^{sa}$, all of whose eigenvalues are in (a,b), and $A = \sum_{i=1}^{l} \lambda_i P_i$ be the spectral decomposition with distinct eigenvalues $\lambda_1, \dots, \lambda_l$ in (a,b). For each $H_1, H_2, \dots, H_r \in M_n^{sa}$ we define

$$
\begin{aligned}
&f^{[r]}(A) \circ (H_1, H_2, \dots, H_r) \\
&:= \sum_{\sigma \in S_r} \sum_{i_1, \dots, i_{r+1}=1}^{l} f^{[r]}(\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{r+1}}) P_{i_1} H_{\sigma(1)} P_{i_2} H_{\sigma(2)} \cdots P_{i_r} H_{\sigma(r)} P_{i_{r+1}},
\end{aligned}
$$

where S_r is the set of all permutations on $\{1, \dots, r\}$. In particular, note ([3, V.3.3]) that if $f \in C^1(a,b)$ and $A = U\mathrm{diag}(\lambda_1, \dots, \lambda_n)U^*$ is a diagonalization, then

$$
\left.\frac{d}{dt}\right|_{t=0} f(A + tH_1) = f^{[1]}(A) \circ H_1 = U\Big(\Big[f^{[1]}(\lambda_i, \lambda_j)\Big]_{ij} \circ U^* H_1 U\Big)U^*,
$$

where $\circ$ stands for the Schur product. The next lemma can be shown in an essentially same way as in the proof of [3, V.3.3].

Lemma 1.3. *Let $A, H_1, \dots, H_m \in M_n^{sa}$ and set $G(x) := A + \sum_{k=1}^{m} x_k H_k$ for $x = (x_1, \dots, x_m) \in \mathbf{R}^m$. Let f be a real-valued C^r function on (a,b) for some $r \in \mathbf{N}$. If the eigenvalues of $G(x)$ are in (a,b) for all x in an open domain D of $\mathbf{R}^m$, then the function $\mathrm{Tr}_n(f(G(x)))$ is C^r on D and*

$$
\begin{aligned}
\frac{\partial^r}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_r}} \mathrm{Tr}_n(f(G(x))) &= \mathrm{Tr}_n\Big(f^{[r]}(G(x)) \circ (H_{k_1}, H_{k_2}, \dots, H_{k_r})\Big) \\
&= \mathrm{Tr}_n\Big(\Big((f')^{[r-1]}(G(x)) \circ (H_{k_1}, \dots, H_{k_{r-1}})\Big) H_{k_r}\Big)
\end{aligned}
$$

for all $1 \le k_1, k_2, \dots, k_r \le m$ and $x \in D$. In particular,

$$
\frac{\partial}{\partial x_k} \mathrm{Tr}_n(f(G(x))) = \mathrm{Tr}_n(f'(G(x)) H_k)
$$

for all $1 \le k \le m$ and $x \in D$.

2. Free LSI for measures on R

In this section we will give a supplementary comment to Biane's work [4] on free version of *logarithmic Sobolev inequality* (*LSI* for short) for measures on $\mathbf{R}$. LSI's were first interested in constructive quantum field theory, and it was Gross [12] who first presented in full generality an LSI for Gaussian measures. Among huge contributions to the topic, Bakry and Emery [1] gave a simple "local" criterion, the so-called Bakry and Emery criterion (see (1.18)), for a given measure to satisfy an LSI. Let M be an m-dimensional smooth complete Riemannian manifold with the volume measure dx. The precise statement that Bakry and Emery established is as follows:

Theorem 2.1. (Bakry and Emery [1]) *Let $\Psi \in C^2(M)$, and set $d\nu(x) := \frac{1}{Z} e^{-\Psi(x)} dx$ with a normalization constant Z. Assume that the Bakry and Emery criterion $\mathrm{Ric}(M) + \mathrm{Hess}(\Psi) \ge \rho I_m$ holds with a constant $\rho > 0$. Then, for every $\mu \in \mathcal{M}(M)$ absolutely continuous with respect to ν one has*

$$
S(\mu, \nu) \le \frac{1}{2\rho} \int_M \left\| \nabla \log \frac{d\mu}{d\nu} \right\|^2 d\mu, \tag{2.1}
$$

whenever the density $d\mu/d\nu$ is smooth on M.

Recall that the left-hand side of (2.1) is the relative entropy (1.1), while the integral in the right-hand side is nothing but the (classical) *relative Fisher information* of μ relative to ν.

Motivated by and based on this theorem, the following "free LSI" was shown by Biane:

Theorem 2.2. (Biane [4]) *Assume that Q is a real-valued C^1 function on $\mathbf{R}$ such that $Q(x) - \frac{\rho}{2}x^2$ is convex on $\mathbf{R}$ with a constant $\rho > 0$. Then, for every $\mu \in \mathcal{M}(\mathbf{R})$ one has*

$$\widetilde{\Sigma}_Q(\mu) \le \frac{1}{2\rho}\Phi_Q(\mu). \tag{2.2}$$

Obviously, the above convexity assumption of Q is equivalent to $Q''(x) \ge \rho$ on $\mathbf{R}$ as long as Q is a C^2*function*.

When $Q(x) = \rho x^2/2$ with $\rho > 0$, the relative free entropy $\widetilde{\Sigma}_Q(\mu)$ is given as

$$\widetilde{\Sigma}_Q(\mu) = -\Sigma(\mu) + \frac{\rho}{2}\int_{\mathbf{R}} x^2\,d\mu(x) - \frac{1}{2}\log\rho - \frac{3}{4}$$

and its minimizer is the $(0,1/\rho)$-semicircular distribution $\gamma_{0,2/\sqrt{\rho}}$ (see (1.10)). Thus, in this special case, for any $\mu \in \mathcal{M}(\mathbf{R})$ having the L^3-density p and satisfying $\int_{\mathbf{R}} x^2\,d\mu(x) < +\infty$, the free LSI becomes

$$-\Sigma(\mu) + \frac{\rho}{2}\int_{\mathbf{R}} x^2\,d\mu(x) - \frac{1}{2}\log\rho - \frac{3}{4} \le \frac{1}{2\rho}\left(\Phi(\mu) - 2\rho + \rho^2\int_{\mathbf{R}} x^2\,d\mu(x)\right) \tag{2.3}$$

because of $2\int_{\mathbf{R}}((Hp)(x))xp(x)\,dx = 1$. Indeed, notice

$$\begin{aligned}
&\int_{\mathbf{R}}\left(\int_{\mathbf{R}}\frac{x-t}{(x-t)^2+\varepsilon^2}\,p(t)\,dt\right)xp(x)\,dx \\
&\quad= \int_{\mathbf{R}} p(t)\left(\int_{\mathbf{R}}\left(1 + \frac{t(x-t)-\varepsilon^2}{(x-t)^2+\varepsilon^2}\right)p(x)\,dx\right)dt \\
&\quad= 1 - \int_{\mathbf{R}} tp(t)\left(\int_{\mathbf{R}}\frac{t-x}{(t-x)^2+\varepsilon^2}\,p(x)\,dx\right)dt \\
&\qquad\quad - \int_{\mathbf{R}} p(t)\left(\int_{\mathbf{R}}\frac{\varepsilon^2}{(t-x)^2+\varepsilon^2}\,p(x)\,dx\right)dt
\end{aligned}$$

so that

$$\begin{aligned}
&2\int_{\mathbf{R}}\left(\int_{\mathbf{R}}\frac{x-t}{(x-t)^2+\varepsilon^2}\,p(t)\,dt\right)xp(x)\,dx \\
&\quad= 1 - \int_{\mathbf{R}} p(t)\left(\int_{\mathbf{R}}\frac{\varepsilon^2}{(t-x)^2+\varepsilon^2}\,p(x)\,dx\right)dt\,.
\end{aligned}$$

Letting $\varepsilon \searrow 0$ gives $2\int(Hp(x))xp(x)\,dx = 1$ as long as $p \in L^3(\mathbf{R})$ (see [17, pp. 92–93]). The inequality (2.3) can be rewritten as

$$\chi(\mu) \ge -\frac{1}{2\rho}\Phi(\mu) - \frac{1}{2}\log\rho + \frac{1}{2}\log 2\pi + 1$$

thanks to the formula (1.2). Maximizing the above right-hand side over $\rho > 0$ gives Voiculescu's inequality ([31, Proposition 7.9])

$$\chi(\mu) \ge \frac{1}{2}\log\frac{2\pi e}{\Phi(\mu)}. \tag{2.4}$$

(The last argument is contained in [5, §§7.2].) In this way, the free LSI in Theorem 2.2 for the functions $Q(x) = \rho x^2/2$ with $\rho > 0$ is equivalent to the inequality (2.4).

In [4, Theorem 3.1] Biane proved Theorem 2.2 when both Q and the density of μ are sufficiently smooth, and the proof of the extension to the general case was omitted. It may

be also worth noting that Lemma 1.3 was implicitly used in [4]. The rest of this section is a supplement to Biane's proof, completing the proof of Theorem 2.2

We need the following general technical lemma.

Lemma 2.3. *Let Q and Q_k, $k \in \mathbf{N}$, be real-valued continuous functions on $\mathbf{R}$ satisfying the following two conditions*:

(a) *Q_k converges to Q uniformly in any finite interval*;

(b) *there exists a real-valued continuous function $\widetilde{Q}$ on $\mathbf{R}$ such that*

$$\lim_{|x|\to+\infty} |x| \exp\left(-\varepsilon\widetilde{Q}(x)\right) = 0 \quad \textit{for every } \varepsilon > 0$$

and $Q_k(x) \geq \widetilde{Q}(x)$ for all $k \in \mathbf{N}$ (so $Q(x) \geq \widetilde{Q}(x)$).

Then, the $B(Q_k)$'s and $B(Q)$ are defined as finite real numbers (see §§1.4), *and one has $\lim_{k\to\infty} B(Q_k) = B(Q)$.*

Proof. By the assumption (b) we can apply the large deviation theorem for self-adjoint random matrices associated to the given Q and the Q_k's. Let μ_Q and μ_{Q_k} be the equilibrium measures associated with Q and Q_k, respectively, and $R > 0$ is chosen so that μ_Q is supported in $[-R, R]$. For each $\varepsilon > 0$, thanks to the assumption (a) we can choose k_0 so that $|Q_k(x) - Q(x)| < \varepsilon$ for all $x \in [-R, R]$ and for all $k \geq k_0$. Then for $k \geq k_0$ we have

$$\begin{aligned}
B(Q) &= B(Q; R) \\
&= \lim_{n\to\infty} \frac{1}{n^2} \log \int_{[-R,R]^n} \exp\left(-n\sum_{i=1}^n Q(x_i)\right) \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^n dx_i \\
&\leq \liminf_{n\to\infty} \frac{1}{n^2} \log \int_{[-R,R]^n} \exp\left(-n\sum_{i=1}^n (Q_k(x_i) + \varepsilon)\right) \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^n dx_i \\
&\leq \varepsilon + \liminf_{n\to\infty} \frac{1}{n^2} \log \int_{\mathbf{R}^n} \exp\left(-n\sum_{i=1}^n Q_k(x_i)\right) \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^n dx_i \\
&= \varepsilon + B(Q_k)
\end{aligned}$$

so that $B(Q) \leq \liminf_{k\to\infty} B(Q_k)$ since ε is arbitrary.

In what follows, we will apply some techniques used in [17, §5.5] and [14]. For $\alpha > 0$ define

$$\begin{aligned}
F(x,y) &:= -\log|x-y| + \frac{1}{2}\left(Q(x) + Q(y)\right), \qquad F_\alpha(x,y) := \min\{F(x,y), \alpha\}; \\
F_k(x,y) &:= -\log|x-y| + \frac{1}{2}\left(Q_k(x) + Q_k(y)\right), \quad F_{k,\alpha}(x,y) := \min\{F_k(x,y), \alpha\}.
\end{aligned}$$

Note that the double integrals of $F(x,y)$ and $F_k(x,y)$ with respect to $\mu \in \mathcal{M}(\mathbf{R})$ are the weighted energy integrals $E_Q(\mu)$ and $E_{Q_k}(\mu)$ associated with Q and Q_k, respectively. Since the tightness of (μ_{Q_k}) can be shown as in the proof of [17, 5.5.3], a subsequence $(\mu_{Q_{k(l)}})$ can be chosen so that $\mu_{Q_{k(l)}}$ weakly converges to some $\mu_0 \in \mathcal{M}(\mathbf{R})$ and

$$\begin{aligned}
&\lim_{l\to\infty} \iint_{\mathbf{R}^2} F_{k(l)}(x,y)\, d\mu_{Q_{k(l)}}(x)\, d\mu_{Q_{k(l)}}(y) \\
&\qquad = \liminf_{k\to\infty} \iint_{\mathbf{R}^2} F_k(x,y)\, d\mu_{Q_k}(x)\, d\mu_{Q_k}(y) = \liminf_{k\to\infty} (-B(Q_k)).
\end{aligned}$$

As in the proof of [17, 5.5.2] it is seen that $F_{k,\alpha}(x,y) \to F_\alpha(x,y)$ uniformly as $k \to \infty$ for each $\alpha > 0$. Hence, we have

$$\begin{aligned}
-B(Q) &\le \iint_{\mathbf{R}^2} F(x,y)\,d\mu_0(x)\,d\mu_0(y) \\
&= \sup_{\alpha>0} \iint_{\mathbf{R}^2} F_\alpha(x,y)\,d\mu_0(x)\,d\mu_0(y) \\
&= \sup_{\alpha>0} \lim_{l\to\infty} \iint_{\mathbf{R}^2} F_{k(l),\alpha}(x,y)\,d\mu_{Q_{k(l)}}(x)\,d\mu_{Q_{k(l)}}(y) \\
&\le \lim_{l\to\infty} \iint_{\mathbf{R}^2} F_{k(l)}(x,y)\,d\mu_{Q_{k(l)}}(x)\,d\mu_{Q_{k(l)}}(y) \\
&= \liminf_{k\to\infty} \left(-B\left(Q_k\right)\right),
\end{aligned}$$

where the first inequality comes from that μ_Q is a minimizer of $E_Q(\mu)$ with $-B(Q) = E_Q(\mu_Q)$. Thus $B(Q) \ge \limsup_{k\to\infty} B(Q_k)$ follows. □

Now, let us prove Theorem 2.2 for the general case. Assume that μ has the density $p = d\mu/dx \in L^3(\mathbf{R})$ and moreover that $\Phi_Q(\mu) = 4\int \left(\left(Hp\right)(x) - \frac{1}{2}Q'(x)\right)^2 d\mu(x)$ is finite. Since $Hp \in L^2\left(\mathbf{R},\mu\right)$ by the former assumption, the latter implies $Q' \in L^2\left(\mathbf{R},\mu\right)$ as well.

At first, suppose further that μ is compactly supported. For each $\varepsilon > 0$ choose a non-negative C^∞ function ϕ_ε supported in $[-\varepsilon,\varepsilon]$ with $\int \phi_\varepsilon(x)\,dx = 1$, and consider the convolution $Q_\varepsilon := Q * \phi_\varepsilon$. Then Q_ε's are C^∞ functions, and $Q_\varepsilon \to Q$ and $Q'_\varepsilon \to Q'$ uniformly on each finite interval as $\varepsilon \searrow 0$. (The last assertion is seen because $Q'_\varepsilon = Q' * \phi_\varepsilon$ follows from the C^1 of Q.) The convexity assumption of Q means that

$$\lambda Q\left(x_1\right) + (1-\lambda)Q\left(x_2\right) - Q\left(\lambda x_1 + (1-\lambda)x_2\right) \ge \frac{\rho}{2}\lambda(1-\lambda)\left(x_1 - x_2\right)^2$$

for all $x_1, x_2 \in \mathbf{R}$ and $0 < \lambda < 1$. This implies the same convexity of Q_ε so that $Q''_\varepsilon(x) \ge \rho$ for all $x \in \mathbf{R}$. Define $p_\varepsilon := p * \phi_\varepsilon$ and $\mu_\varepsilon \in \mathcal{M}(\mathbf{R})$ by $d\mu_\varepsilon(x) := p_\varepsilon(x)\,dx$. Moreover, consider $Q_{\mu_\varepsilon}(x) := 2\int_{\mathbf{R}} \log|x-y|\,d\mu_\varepsilon(y)$, which is a C^∞ function on $\mathbf{R}$. Then we have $Q'_{\mu_\varepsilon}(x) = 2(Hp_\varepsilon)(x)$ for a.e. $x \in \mathbf{R}$ (see the proof of Lemma 3.2 (i) in §3). Hence, the proof of Theorem 2.2 in [4] implies that

$$\widetilde{\Sigma}_{Q_\varepsilon}(\mu_\varepsilon) \le \frac{1}{2\rho}\Phi_{Q_\varepsilon}(\mu_\varepsilon) \quad \text{for } \varepsilon > 0. \tag{2.5}$$

Since the convexity assumption of Q implies that $Q_\varepsilon(x) \ge ax^2 + b$ for some $a > 0$ and $b \in \mathbf{R}$, Lemma 2.3 gives

$$\lim_{\varepsilon\searrow 0} B(Q_\varepsilon) = B(Q).$$

Furthermore, notice that $\|p_\varepsilon - p\|_{L^3} \to 0$ and hence $\|Hp_\varepsilon - Hp\|_{L^3} \to 0$ as $\varepsilon \searrow 0$ so that we get

$$\lim_{\varepsilon\searrow 0} \int_{\mathbf{R}} Q_\varepsilon(x)\,d\mu_\varepsilon(x) = \int_{\mathbf{R}} Q(x) d\mu(x),$$

$$\lim_{\varepsilon\searrow 0} \int_{\mathbf{R}} \left(\left(Hp_\varepsilon\right)(x) - \frac{1}{2}Q'_\varepsilon(x)\right)^2 d\mu_\varepsilon(x) = \int_{\mathbf{R}} \left(\left(Hp\right)(x) - \frac{1}{2}Q'(x)\right)^2 d\mu(x).$$

From (2.5) and the above convergences together with the upper semicontinuity of $\Sigma(\mu)$ (see [17, 5.3.2]) we have

$$\widetilde{\Sigma}_Q(\mu) \le \liminf_{\varepsilon\searrow 0} \widetilde{\Sigma}_{Q_\varepsilon}(\mu_\varepsilon) \le \lim_{\varepsilon\searrow 0} \frac{1}{2\rho}\Phi_{Q_\varepsilon}(\mu_\varepsilon) = \frac{1}{2\rho}\Phi_Q(\mu).$$

Next, let us treat the case where μ is not compactly supported. For $R>0$ set $d\mu_R(x) := \frac{1}{\mu([-R,R])}\chi_{[-R,R]}(x)\,d\mu(x)$, whose density is given by $p_R := \frac{1}{\mu([-R,R])}\chi_{[-R,R]}\,p$. Then, $\|p_R - p\|_{L^3}\to 0$ and $\|Hp_R - Hp\|_{L^3}\to 0$ as $R\to+\infty$ so that

$$\lim_{R\to+\infty}\int_{\mathbf{R}}((Hp_R)(x))^2\,p_R(x)\,dx = \int_{\mathbf{R}}((Hp)(x))^2\,p(x)\,dx$$

and

$$\begin{aligned}
&\int_{\mathbf{R}}\left(Q'(x)\right)^2|p_R(x)-p(x)|\,dx \\
&\quad\le \int_{\mathbf{R}\setminus[-R,R]}\left(Q'(x)\right)^2 p(x)\,dx + \left(\frac{1}{\mu\left([-R,R]\right)}-1\right)\int_{\mathbf{R}}\left(Q'(x)\right)^2 p(x)\,dx \\
&\quad\longrightarrow 0 \quad \text{as } R\to+\infty.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\left|\int_{\mathbf{R}}(Hp_R)(x)Q'(x)\,d\mu_R(x) - \int_{\mathbf{R}}(Hp)(x)Q'(x)\,d\mu(x)\right| \\
&\le \left\{\int_{\mathbf{R}}\left((Hp_R)(x)\left(\frac{p_R(x)}{p(x)}-1\right)\right)^2 p(x)\,dx\right\}^{1/2}\left(\int_{\mathbf{R}}Q'(x)^2p(x)\,dx\right)^{1/2} \\
&\qquad + \left(\int_{\mathbf{R}}((Hp_R)(x)-(Hp)(x))^2\,p(x)\,dx\right)^{1/2}\left(\int_{\mathbf{R}}Q'(x)^2p(x)\,dx\right)^{1/2} \\
&\le \left\{\int_{\mathbf{R}\setminus[-R,R]}((Hp_R)(x))^2\,p(x)\,dx + \left(\frac{1}{\mu\left([-R,R]\right)}-1\right)^2\int_{\mathbf{R}}((Hp_R)(x))^2\,p(x)\,dx\right\}^{1/2} \\
&\qquad\qquad \times\left(\int_{\mathbf{R}}Q'(x)^2p(x)\,dx\right)^{1/2} \\
&\qquad + \left(\int_{\mathbf{R}}|(Hp_R)(x)-(Hp)(x)|^3\,dx\right)^{1/3}\left(\int_{\mathbf{R}}p(x)^3\,dx\right)^{1/6}\left(\int_{\mathbf{R}}Q'(x)^2p(x)\,dx\right)^{1/2} \\
&\longrightarrow 0 \quad \text{as } R\to+\infty.
\end{aligned}$$

In the above, the first inequality is obtained by the Cauchy-Schwarz inequality with respect to $d\mu(x)=p(x)dx$ and the second one is by the Hölder inequality with respect to dx. From the above convergences we get

$$\lim_{R\to+\infty}\Phi_Q\left(\mu_Q\right) = \Phi_Q(\mu). \tag{2.6}$$

On the other hand, we get

$$\widetilde{\Sigma}_Q(\mu) \le \liminf_{R\to+\infty}\widetilde{\Sigma}_Q(\mu_R) \tag{2.7}$$

thanks to the monotone convergence theorem and the upper semicontinuity of $\Sigma(\mu)$. Therefore, the desired inequality follows from (2.6), (2.7) and the first case of μ being compactly supported. □

3. Free LSI for measures on T

In this section we will proceed to the free analog of logarithmic Sobolev inequalities for measures on $\mathbf{T}$. The idea here is essentially same as Biane's work [4] mentioned in §2. Namely, the free analog arises as the scaling limit in the scale $1/n^2$ of the classical one (2.1) on the special unitary group $\mathrm{SU}(n)$. However, there is an essential difference between his argument and ours; we need full power of large deviation principle (especially the weak convergence

of the empirical eigenvalue distribution to the equilibrium measure almost surely), while the weak convergence of the mean eigenvalue distribution is enough in the proof of [4, Theorem 3.1].

Let us start with some lemmas.

Lemma 3.1. *Let Q be a harmonic function on a neighborhood of the unit disk $\{\zeta \in \mathbf{C} : |\zeta| \le 1\}$. For each $n \in \mathbf{N}$ and each $U \in \mathrm{SU}(n)$ define $Q(U)$ via the functional calculus and set $\Psi(U) := \mathrm{Tr}_n(Q(U))$. Then one has*

(i) *The function $\Psi(U)$ on $\mathrm{SU}(n)$ is C^∞.*

(ii) $\nabla\Psi(U) = \sqrt{-1}\left(Q'(U) - \frac{1}{n}\mathrm{Tr}_n(Q'(U))I_n\right)$.

(iii) *If $Q\left(e^{\sqrt{-1}t}\right) - \frac{\rho}{2}t^2$ is convex on $\mathbf{R}$ for some constant $\rho > 0$, then* $\mathrm{Hess}(\Psi) \ge \rho I_{n^2-1}$.

Proof. Set $f(t) := Q\left(e^{\sqrt{-1}t}\right)$ for $t \in \mathbf{R}$, and let $Y_k := \sqrt{-1}X_k$ with $X_k = X_k^*$, $1 \le k \le n^2-1$, be a basis of the Lie algebra $\mathfrak{su}(n) = \{T \in M_n(\mathbf{C}) : T + T^* = 0,\ \mathrm{Tr}_n(T) = 0\}$ ($\cong \mathbf{R}^{n^2-1}$). For any $U_0 = e^{\sqrt{-1}A_0} \in \mathrm{SU}(n)$ with $\sqrt{-1}A_0 \in \mathfrak{su}(n)$ and for $x = (x_1, \dots, x_{n^2-1}) \in \mathbf{R}^{n^2-1}$, we write

$$\Psi\left(\exp\left(\sqrt{-1}A_0 + \sum_{k=1}^{n^2-1} x_k Y_k\right)\right) = \mathrm{Tr}_n\left(f\left(A_0 + \sum_{k=1}^{n^2-1} x_k X_k\right)\right).$$

The C^∞ of f on $\mathbf{R}$ immediately follows from the assumption of Q. In fact, for each $t_0 \in \mathbf{R}$, the function $f(t_0+t)$ has a power series expansion for t near 0. Hence, thanks to Lemma 1.3 we have (i) and

$$\begin{aligned}
\nabla\Psi(U_0) &= \sum_{k=1}^{n^2-1} \mathrm{Tr}_n(f'(A_0)Y_k)Y_k \\
&= \sum_{k=1}^{n^2-1} \mathrm{Tr}_n\left(\left(f'(A_0) - \frac{1}{n}\mathrm{Tr}_n(f'(A_0))I_n\right)Y_k\right)Y_k \\
&= \sum_{k=1}^{n^2-1} \left\langle \sqrt{-1}\left(f'(A_0) - \frac{1}{n}\mathrm{Tr}_n(f'(A_0))I_n\right), Y_k \right\rangle_{\mathrm{Tr}_n} Y_k \\
&= \sqrt{-1}\left(f'(A_0) - \frac{1}{n}\mathrm{Tr}_n(f'(A_0))I_n\right) \\
&= \sqrt{-1}\left(Q'(U_0) - \frac{1}{n}\mathrm{Tr}_n(Q'(U_0))I_n\right),
\end{aligned}$$

implying (ii).

Set $F(t) := Q\left(e^{\sqrt{-1}t}\right) - \frac{\rho}{2}t^2$ for $t \in \mathbf{R}$. For any $U_0 = e^{\sqrt{-1}A_0} \in \mathrm{SU}(n)$ with $\sqrt{-1}A_0 \in \mathfrak{su}(n)$ and for $(x_1, \dots, x_{n^2-1}) \in \mathbf{R}^{n^2-1}$, we have

$$\begin{aligned}
&\Psi\left(\exp\left(\sqrt{-1}A_0 + \sum_{k=1}^{n^2-1} x_k Y_k\right)\right) \\
&\quad = \mathrm{Tr}_n\left(F\left(A_0 + \sum_{k=1}^{n^2-1} x_k X_k\right)\right) + \frac{\rho}{2}\mathrm{Tr}_n\left(\left(A_0 + \sum_{k=1}^{n^2-1} x_k X_k\right)^2\right) \\
&\quad = \mathrm{Tr}_n\left(F\left(A_0 + \sum_{k=1}^{n^2-1} x_k X_k\right)\right) + \frac{\rho}{2}\mathrm{Tr}_n(A_0^2) + \rho\sum_{k=1}^{n^2-1} \mathrm{Tr}_n(A_0X_k)x_k + \frac{\rho}{2}\sum_{k=1}^{n^2-1} x_k^2.
\end{aligned}$$

Since $F(t)$ is convex on $\mathbf{R}$, it is known ([24, 3.1]) that $\mathrm{Tr}_n\bigl(F(A_0+\sum_{k=1}^{n^2-1}x_kX_k)\bigr)$ is convex in $(x_1,\dots,x_{n^2-1})$ so that (iii) follows. $\square$

Lemma 3.2. *Assume that $\mu\in\mathcal{M}(\mathbf{T})$ has a continuous density $p=d\mu/d\zeta$ and that $Q_\mu(\zeta):=2\int_{\mathbf{T}}\log|\zeta-\eta|\,d\mu(\eta)$ is C^1 on $\mathbf{T}$. Then one has*

(i) *$Q_\mu'(\zeta)=(Hp)(\zeta)$ for a.e. $\zeta\in\mathbf{T}$;*

(ii) *$\int_{\mathbf{T}}((Hp)(\zeta))\,p(\zeta)\,d\zeta=0$.*

Proof. (i) Let f be an arbitrary C^1 function on $\mathbf{T}$. Then we have

$$\begin{aligned}
&\int_0^{2\pi}\frac{d}{d\theta}Q_\mu\left(e^{\sqrt{-1}\theta}\right)f\left(e^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi}\\
&\quad=-\int_0^{2\pi}Q_\mu\left(e^{\sqrt{-1}\theta}\right)\frac{d}{d\theta}f\left(e^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi}\\
&\quad=-\lim_{\varepsilon\searrow0}\int_{|\theta-t|\ge\varepsilon}2\log\left|e^{\sqrt{-1}\theta}-e^{\sqrt{-1}t}\right|\frac{d}{d\theta}f\left(e^{\sqrt{-1}\theta}\right)p\left(e^{\sqrt{-1}t}\right)\frac{d\theta\times dt}{(2\pi)^2}\\
&\quad=-\lim_{\varepsilon\searrow0}\int_0^{2\pi}\left(\int_{|\theta-t|\ge\varepsilon}\log\left(2(1-\cos(\theta-t))\right)\frac{d}{d\theta}f\left(e^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi}\right)p\left(e^{\sqrt{-1}t}\right)\frac{dt}{2\pi},
\end{aligned}$$

where the second equality is due to the fact that $\log\left|e^{\sqrt{-1}\theta}-e^{\sqrt{-1}t}\right|\frac{d}{d\theta}f\left(e^{\sqrt{-1}\theta}\right)$ is bounded above. Integrating by parts we get

$$\begin{aligned}
&\int_{|\theta-t|\ge\varepsilon}\log\left(2(1-\cos(\theta-t))\right)\frac{d}{d\theta}f\left(e^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi}\\
&\quad=-\frac{\log\left(2\left(1-\cos\varepsilon\right)\right)}{2\pi}\left(f\left(e^{\sqrt{-1}(t+\varepsilon)}\right)-f\left(e^{\sqrt{-1}(t-\varepsilon)}\right)\right)-\int_{|\theta-t|\ge\varepsilon}\frac{f\left(e^{\sqrt{-1}\theta}\right)}{\tan\left(\frac{\theta-t}{2}\right)}\frac{d\theta}{2\pi},
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_0^{2\pi}\frac{d}{d\theta}Q_\mu\left(e^{\sqrt{-1}\theta}\right)f\left(e^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi}\\
&\quad=\lim_{\varepsilon\searrow0}\Biggl\{\frac{\log\left(2\left(1-\cos\varepsilon\right)\right)}{2\pi}\int_0^{2\pi}\left(f\left(e^{\sqrt{-1}(t+\varepsilon)}\right)-f\left(e^{\sqrt{-1}(t-\varepsilon)}\right)\right)p\left(e^{\sqrt{-1}t}\right)\frac{dt}{2\pi}\\
&\qquad\qquad+\int_0^{2\pi}\left(\int_{|\theta-t|\ge\varepsilon}\frac{f\left(e^{\sqrt{-1}\theta}\right)}{\tan\left(\frac{\theta-t}{2}\right)}\frac{d\theta}{2\pi}\right)p\left(e^{\sqrt{-1}t}\right)\frac{dt}{2\pi}\Biggr\}\\
&\quad=\lim_{\varepsilon\searrow0}\int_0^{2\pi}\left(\int_{|\theta-t|\ge\varepsilon}\frac{p\left(e^{\sqrt{-1}t}\right)}{\tan\left(\frac{\theta-t}{2}\right)}\frac{dt}{2\pi}\right)f\left(e^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi}\\
&\quad=\int_0^{2\pi}(Hp)\left(e^{\sqrt{-1}\theta}\right)f\left(e^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi}.
\end{aligned}$$

In the above, the second equality comes from $\left|f\left(e^{\sqrt{-1}(t+\varepsilon)}\right)-f\left(e^{\sqrt{-1}(t-\varepsilon)}\right)\right|=O(\varepsilon)$ uniformly for $t\in[0,2\pi)$, and since we have in particular $p\in L^2(\mathbf{T})$, the last one does from the L^2-convergence of the involved principle value integral to Hp (see [11, 12.8.2 (2)]). Hence, the desired assertion follows since f is arbitrary.

(ii) is seen by taking the limit as $\varepsilon \searrow 0$ of

$$\int_0^{2\pi} \left(\int_{|t-\theta|\ge\varepsilon} \frac{p\left(e^{\sqrt{-1}t}\right)}{\tan\left(\frac{\theta-t}{2}\right)} \frac{dt}{2\pi} \right) p\left(e^{\sqrt{-1}\theta}\right) \frac{d\theta}{2\pi}$$
$$= - \int_0^{2\pi} \left(\int_{|\theta-t|\ge\varepsilon} \frac{p\left(e^{\sqrt{-1}\theta}\right)}{\tan\left(\frac{t-\theta}{2}\right)} \frac{d\theta}{2\pi} \right) p\left(e^{\sqrt{-1}t}\right) \frac{dt}{2\pi}$$

thanks to the L^2-convergence of the principle value integral as mentioned above. □

Theorem 3.3. *Let Q be a real-valued C^1 function on $\mathbf{T}$ such that $Q\left(e^{\sqrt{-1}t}\right) - \frac{\rho}{2}t^2$ is convex on $\mathbf{R}$ with a constant $\rho > -1/2$. Then, for every $\mu \in \mathcal{M}(\mathbf{T})$ one has*

$$\widetilde{\Sigma}_Q(\mu) \le \frac{1}{1+2\rho} F_Q(\mu). \tag{3.1}$$

In the special case where $Q \equiv 0$ and $\rho = 0$, the above (3.1) becomes

$$-\Sigma(\mu) \le F(\mu)$$

and the equilibrium measure μ_Q is the uniform distribution $d\zeta$.

In particular, the theorem implies that $F_Q(\mu) \ge 0$; that is,

$$\int_{\mathbf{T}} \left((Hp)(\zeta) - Q'(\zeta)\right)^2 d\mu(\zeta) \ge \left(\int_{\mathbf{T}} Q'(\zeta)\, d\mu(\zeta) \right)^2$$

for every $\mu \in \mathcal{M}(\mathbf{T})$ under the above assumption of Q. Also, suppose that the equilibrium measure μ_Q has a continuous density and its support is $\mathbf{T}$; then we have $Q(\zeta) = 2\int_{\mathbf{T}} \log|\zeta - \eta|\, d\mu_Q(\eta)$ for all $\zeta \in \mathbf{T}$ due to [26, Theorem I.3.1] so that Lemma 3.2 gives $F_Q(\mu_Q) = 0$.

Proof of Theorem 3.3. First, let us assume:

(a) Q is harmonic on a neighborhood of the unit disk;
(b) μ has a continuous density $p = d\mu/d\zeta$, and $Q_\mu(\zeta) := 2\int_{\mathbf{T}} \log|\zeta - \eta|\, d\mu(\eta)$ is harmonic on a neighborhood of the unit disk.

For each $n \in \mathbf{N}$ define $n \times n$ special unitary random matrices $\lambda_n^{\mathrm{SU}}(Q)$ and $\lambda_n^{\mathrm{SU}}(Q_\mu)$ as in (1.16), i.e.,

$$d\lambda_n^{\mathrm{SU}}(Q)(U) := \frac{1}{Z_n^{\mathrm{SU}}(Q)} \exp(-n\mathrm{Tr}_n(Q(U)))\, dU,$$
$$d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) := \frac{1}{Z_n^{\mathrm{SU}}(Q_\mu)} \exp(-n\mathrm{Tr}_n(Q_\mu(U)))\, dU.$$

Let $\tilde{\lambda}_n^{\mathrm{SU}}(Q)$ and $\tilde{\lambda}_n^{\mathrm{SU}}(Q_\mu)$ be their joint eigenvalue distributions on $\mathbf{T}^{n-1}$. Also, let $\hat{\lambda}_n^{\mathrm{SU}}(Q)$ and $\hat{\lambda}_n^{\mathrm{SU}}(Q_\mu)$ be their mean eigenvalue distributions (see §§1.6). According to Theorem 1.2, the empirical eigenvalue distribution of $\lambda_n^{\mathrm{SU}}(Q_\mu)$ satisfies the large deviation principle in the scale $1/n^2$ whose rate functions is $\widetilde{\Sigma}_{Q_\mu}(\mu)$. Moreover, note ([26, Theorem I.3.1]) that the equilibrium measure associated with Q_μ (or the minimizer of $\widetilde{\Sigma}_{Q_\mu}$) is the given μ. This large deviation principle guarantees the following facts (i) and (ii), which will be the key ingredients in our arguments below.

(i) $\hat{\lambda}_n^{\mathrm{SU}}(Q_\mu) \to \mu$ weakly as $n \to \infty$;
(ii) the empirical distribution $\frac{1}{n}(\zeta_1 + \cdots + \zeta_n)$ weakly converges to μ almost surely as $n \to \infty$ when $(\zeta_1, \ldots, \zeta_{n-1})$ is distributed according to $\tilde{\lambda}_n^{\mathrm{SU}}(Q_\mu)$ and $\zeta_n = (\zeta_1 \cdots \zeta_{n-1})^{-1}$.

Set $\Psi_n(U) := n\mathrm{Tr}_n(Q(U))$ for $U \in \mathrm{SU}(n)$. Lemma 3.1 (iii) and (1.19) verify the Bakry and Emery criterion:

$$\mathrm{Ric}(\mathrm{SU}(n)) + \mathrm{Hess}(\Psi_n) \ge \left(\frac{n}{2} + n\rho\right) I_{n^2-1}. \tag{3.2}$$

Thus, by Theorem 2.1 due to Bakry and Emery we get

$$S\big(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\big) \le \frac{1}{2\left(\frac{n}{2} + n\rho\right)} \int_{\mathrm{SU}(n)} \left\|\nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}\right\|_{HS}^2 d\lambda_n^{\mathrm{SU}}(Q_\mu). \tag{3.3}$$

Notice

$$\frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U) = \frac{\widetilde{Z}_n^{\mathrm{SU}}(Q)}{\widetilde{Z}_n^{\mathrm{SU}}(Q_\mu)} \exp\big(-n\mathrm{Tr}_n(Q_\mu(U)) + n\mathrm{Tr}_n(Q(U))\big), \quad U \in \mathrm{SU}(n), \tag{3.4}$$

where $\widetilde{Z}_n^{\mathrm{SU}}(Q)$ and $\widetilde{Z}_n^{\mathrm{SU}}(Q_\mu)$ are the normalization constants of the joint eigenvalue distributions (see §§1.6). Hence, we have

$$\begin{aligned}
&\frac{1}{n^2} S\big(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\big) \\
&\quad = \frac{1}{n^2} \int_{\mathrm{SU}(n)} \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}(Q)}(U)} \, d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\
&\quad = \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q) - \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q_\mu) \\
&\qquad - \int_{\mathrm{SU}(n)} \frac{1}{n} \mathrm{Tr}_n(Q_\mu(U)) \, d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) + \int_{\mathrm{SU}(n)} \frac{1}{n} \mathrm{Tr}_n(Q(U)) \, d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\
&\quad = \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q) - \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q_\mu) \\
&\qquad - \int_{\mathbf{T}} Q_\mu(\zeta) \, d\hat{\lambda}_n^{\mathrm{SU}}(Q_\mu)(\zeta) + \int_{\mathbf{T}} Q(\zeta) \, d\hat{\lambda}_n^{\mathrm{SU}}(Q_\mu)(\zeta),
\end{aligned}$$

and therefore, thanks to (b) and (i) above,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^2} S\big(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\big) \\
&\qquad = B(Q) - B(Q_\mu) - \int_{\mathbf{T}} Q_\mu(\zeta) \, d\mu(\zeta) + \int_{\mathbf{T}} Q(\zeta) \, d\mu(\zeta) = \widetilde{\Sigma}_Q(\mu),
\end{aligned} \tag{3.5}$$

where the last equality comes from that μ is the minimizer with $\widetilde{\Sigma}_{Q_\mu}(\mu) = 0$, i.e.,

$$\int_{\mathbf{T}} Q_\mu(\zeta) \, d\mu(\zeta) + B(Q_\mu) = \Sigma(\mu).$$

Therefore, the scaling limit in the scale $1/n^2$ of the left-hand side of (3.3) becomes the relative free entropy $\widetilde{\Sigma}_Q(\mu)$. We will seek for the scaling limit in the scale $1/n^2$ of the right-hand side of (3.3). By (3.4) and Lemma 3.1 (ii), we have

$$\begin{aligned}
\nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U) &= -n\nabla\big(\mathrm{Tr}_n(Q_\mu(U)) - \mathrm{Tr}_n(Q(U))\big) \\
&= -\sqrt{-1}\Big\{n\big(Q'_\mu(U) - Q'(U)\big) - \big(\mathrm{Tr}_n\big(Q'_\mu(U) - Q'(U)\big)\big) I_n\Big\}
\end{aligned}$$

so that

$$\begin{aligned}
&\left\|\nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U)\right\|_{HS}^2 \\
&\qquad = n^2 \mathrm{Tr}_n\Big(\big(Q'_\mu(U) - Q'(U)\big)^2\Big) - n\Big(\mathrm{Tr}_n\big(Q'_\mu(U) - Q'(U)\big)\Big)^2.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\frac{1}{n^2}\cdot\frac{1}{2\left(\frac{n}{2}+n\rho\right)}\int_{\mathrm{SU}(n)}\left\|\nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U)\right\|_{HS}^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\
&\quad = \frac{1}{1+2\rho}\Bigg\{\int_{\mathrm{SU}(n)}\frac{1}{n}\mathrm{Tr}_n\Big(\big(Q'_\mu(U)-Q'(U)\big)^2\Big)\,d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\
&\qquad\qquad - \int_{\mathrm{SU}(n)}\frac{1}{n^2}\Big(\mathrm{Tr}_n\big(Q'_\mu(U)-Q'(U)\big)\Big)^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U)\Bigg\}.
\end{aligned}$$

The above-mentioned fact (i) implies that

$$\begin{aligned}
&\int_{\mathrm{SU}(n)}\frac{1}{n}\mathrm{Tr}_n\Big(\big(Q'_\mu(U)-Q'(U)\big)^2\Big)\,d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\
&\qquad = \int_{\mathbf{T}}\big(Q'_\mu(\zeta)-Q'(\zeta)\big)^2\,d\hat{\lambda}_n^{\mathrm{SU}}(Q_\mu)(\zeta) \\
&\qquad \longrightarrow \int_{\mathbf{T}}\big(Q'_\mu(\zeta)-Q'(\zeta)\big)^2\,d\mu(\zeta) \quad \text{as } n\to\infty,
\end{aligned}$$

while the above fact (ii) does that

$$\begin{aligned}
&\int_{\mathrm{SU}(n)}\frac{1}{n^2}\Big(\mathrm{Tr}_n\big(Q'_\mu(U)-Q'(U)\big)\Big)^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\
&\quad = \int_{\mathbf{T}^{n-1}}\left(\frac{1}{n}\sum_{i=1}^n\big(Q'_\mu(\zeta_i)-Q'(\zeta_i)\big)\right)^2 d\tilde{\lambda}_n^{\mathrm{SU}}(Q_\mu)(\zeta_1,\dots,\zeta_{n-1}) \\
&\qquad\qquad\qquad\qquad \text{with } \zeta_n := (\zeta_1\cdots\zeta_{n-1})^{-1} \\
&\quad \longrightarrow \left(\int_{\mathbf{T}}\big(Q'_\mu(\zeta)-Q'(\zeta)\big)\,d\mu(\zeta)\right)^2 \quad \text{as } n\to\infty
\end{aligned}$$

Thanks to the assumption (b), Lemma 3.2 implies that

$$\begin{aligned}
\left(\int_{\mathbf{T}}\big(Q'_\mu(\zeta)-Q'(\zeta)\big)\,d\mu(\zeta)\right)^2 &= \left(\int_{\mathbf{T}}((Hp)(\zeta))p(\zeta)\,d\zeta - \int_{\mathbf{T}}Q'(\zeta)\,d\mu(\zeta)\right)^2 \\
&= \left(\int_{\mathbf{T}}Q'(\zeta)\,d\mu(\zeta)\right)^2
\end{aligned}$$

so that we get

$$\lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{1}{2\left(\frac{n}{2}+n\rho\right)}\int_{\mathrm{SU}(n)}\left\|\nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U)\right\|_{HS}^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) = \frac{1}{1+2\rho}F_Q(\mu). \tag{3.6}$$

By (3.3), (3.5) and (3.6) we have shown the desired inequality (3.1) under the assumptions (a) and (b).

Next, let us deal with a general Q as stated in the theorem. Let $\mu \in \mathcal{M}(\mathbf{T})$ with a density $p = d\mu/d\zeta \in L^3(\mathbf{T})$. For each $0 < r < 1$, we consider the Poisson integrals Q_r and p_r of Q and p, respectively; that is,

$$\begin{aligned} Q_r\left(e^{\sqrt{-1}\theta}\right) &:= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) Q\left(e^{\sqrt{-1}t}\right) dt, \\ p_r\left(e^{\sqrt{-1}\theta}\right) &:= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) p\left(e^{\sqrt{-1}t}\right) dt \end{aligned}$$

with the Poisson kernel $P_r(\theta) := (1-r^2)/(1-2r\cos\theta + r^2)$. Define $\mu_r \in \mathcal{M}(\mathbf{T})$ by $d\mu_r(\zeta) := p_r(\zeta)d\zeta$. Then it is plain to see that Q_r satisfies the assumption (a) and that μ_r does (b). The convexity assumption of Q in the theorem means that

$$\lambda Q\left(e^{\sqrt{-1}s}\right) + (1-\lambda) Q\left(e^{\sqrt{-1}t}\right) - Q\left(e^{\sqrt{-1}(\lambda s + (1-\lambda)t)}\right) \geq \frac{\rho}{2}\lambda(1-\lambda)(t-s)^2$$

for all $s, t \in \mathbf{R}$ and $0 < \lambda < 1$. It is easy to check that each Q_r, $0 < r < 1$, satisfies the same convexity assumption so that

$$\widetilde{\Sigma}_{Q_r}(\mu) \leq \frac{1}{1+2\rho} F_{Q_r}(\mu) \tag{3.7}$$

by what we have already shown. It is known (see [16] and also [17, p.224]) that $\mu_r \to \mu$ weakly and $\Sigma(\mu_r) \to \Sigma(\mu)$ as $r \nearrow 1$. Moreover, it is known (see [19, 5.3.2]) that $\|Q_r - Q\|_\infty \to 0$ as $r \nearrow 1$, where $\|\cdot\|_\infty$ means the uniform norm on $C(\mathbf{T})$. Since it is easily seen that

$$\left| \frac{1}{n^2} \log \tilde{Z}_n(Q_r) - \frac{1}{n^2} \log \tilde{Z}_n(Q) \right| \leq \|Q_r - Q\|_\infty ,$$

we have $B(Q_r) \to B(Q)$ as $r \nearrow 1$. Therefore, we get

$$\lim_{r \nearrow 1} \widetilde{\Sigma}_{Q_r}(\mu) = \widetilde{\Sigma}_Q(\mu).$$

Notice that $\|p_r - p\|_{L^3} \to 0$ and hence $\|Hp_r - Hp\|_{L^3} \to 0$ as $r \nearrow 1$. Since Q is a C^1 function, Q_r' becomes the Poisson integral of Q' so that $\|Q_r' - Q'\|_\infty \to 0$ as $r \nearrow 1$ as well. These imply that

$$\begin{aligned} \lim_{r \nearrow 1} F_{Q_r}(\mu) &= \lim_{r \nearrow 1} \left\{ \int_{\mathbf{T}} \left((Hp_r)(\zeta) - Q_r'(\zeta)\right)^2 d\mu_r(\zeta) - \left(\int_{\mathbf{T}} Q_r'(\zeta)\, d\mu_r(\zeta)\right)^2 \right\} \\ &= \int_{\mathbf{T}} \left((Hp)(\zeta) - Q'(\zeta)\right)^2 d\mu(\zeta) - \left(\int_{\mathbf{T}} Q'(\zeta) d\mu(\zeta)\right)^2 = F_Q(\mu). \end{aligned}$$

Hence, the desired inequality (3.1) follows by taking the limit of (3.7). □

4. Free TCI for measures on R

The second aim of this paper is to obtain the free analog of transportation cost inequalities for measures on $\mathbf{R}$ and on $\mathbf{T}$. We deal with probability measures on $\mathbf{R}$ in this section and those on $\mathbf{T}$ in the next section. The (classical) transportation cost inequalities compare the Wasserstein distance with the relative entropy (see (1.1)) for two probability measures. Let us first recall the definition of the Wasserstein distance. Let $\mathcal{X}$ be a Polish space with a metric d. The (quadratic) *Wasserstein distance* between $\mu, \nu \in \mathcal{M}(\mathcal{X})$ is defined by

$$W(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \sqrt{\iint_{\mathcal{X} \times \mathcal{X}} \frac{1}{2} d(x,y)^2 \, d\pi(x,y)}, \tag{4.1}$$

where $\Pi(\mu,\nu)$ denotes the set of all probability measures on $\mathcal{X}\times\mathcal{X}$ with marginals μ and ν, i.e., $\pi(\,\cdot\times\mathcal{X})=\mu$ and $\pi(\mathcal{X}\times\cdot\,)=\nu$. The Wasserstein distance is sometimes defined with the integral of $d(x,y)^2$ instead of $\frac{1}{2}d(x,y)^2$. The next lemma is well known and easy to show.

Lemma 4.1. *$W(\mu,\nu)$ is weakly lower semicontinuous in $\mu,\nu\in\mathcal{M}(\mathcal{X})$; namely, if $\mu_n,\nu_n\in\mathcal{M}(\mathcal{X})$, $\mu_n\to\mu$ and $\nu_n\to\nu$ in the weak topology, then*

$$W(\mu,\nu)\le\liminf_{n\to\infty}W(\mu_n,\nu_n).$$

In the typical case where $\mathcal{X}=\mathbf{R}^n$ and $d(x,y)=\|x-y\|$, the usual Euclidean metric, let g_n be the standard Gaussian measure, i.e., $dg_n(x):=(2\pi)^{-n/2}e^{-\|x\|^2/2}\,dx$ (dx means the Lebesgue measure on $\mathbf{R}^n$). The celebrated *transportation cost inequality* (*TCI* for short) of Talagrand [28] is

$$W(\mu,g_n)\le\sqrt{S(\mu,g_n)},\qquad \mu\in\mathcal{M}(\mathbf{R}^n).$$

This inequality is a bit extended as follows (see [21]):

Theorem 4.2. *Let $\Psi:\mathbf{R}^n\to\mathbf{R}$ and assume that $\Psi(x)-\frac{\rho}{2}\|x\|^2$ is convex on $\mathbf{R}^n$ with a constant $\rho>0$. If $d\nu(x):=\frac{1}{Z}e^{-\Psi(x)}\,dx\in\mathcal{M}(\mathbf{R}^n)$ with a normalization constant Z, then*

$$W(\mu,\nu)\le\sqrt{\frac{1}{\rho}S(\mu,\nu)},\qquad \mu\in\mathcal{M}(\mathbf{R}^n).$$

In [25] Otto and Villani established the interrelation between LSI and TCI by a technique using partial differential equations. Their result, combined with Bakry and Emery's LSI ([1] or Theorem 2.1), implies the following TCI in a setup on Riemannian manifolds, which will play a crucial role in deriving our free analog of TCI for measures on $\mathbf{T}$. In the theorem, let M be an m-dimensional smooth complete Riemannian manifold equipped with the geodesic distance $d(x,y)$ and the volume measure dx.

Theorem 4.3. (Bakry and Emery [1] and Otto and Villani [25]) *Let Ψ be a real-valued C^2 function on M and set $d\nu(x):=\frac{1}{Z}e^{-\Psi(x)}\,dx\in\mathcal{M}(M)$ with a normalization constant Z. If the Bakry and Emery criterion* $\mathrm{Ric}(M)+\mathrm{Hess}(\Psi)\ge\rho I_m$ *holds with a constant $\rho>0$, then*

$$W(\mu,\nu)\le\sqrt{\frac{1}{\rho}S(\mu,\nu)},\qquad \mu\in\mathcal{M}(M).$$

On the other hand, the following free analog of Talagrand's TCI is shown by Biane and Voiculescu [6]. Recall that $\gamma_{0,2}$ is the standard semicircular measure (see (1.10)).

Theorem 4.4. (Biane and Voiculescu [6]) *For every compactly supported $\mu\in\mathcal{M}(\mathbf{R})$,*

$$W(\mu,\gamma_{0,2})\le\sqrt{-\Sigma(\mu)+\int\frac{x^2}{2}\,d\mu(x)-\frac{3}{4}}. \tag{4.2}$$

In the rest of this section we will present a new proof of the above free TCI in a more general situation by using a random matrix technique. In fact, the classical TCI on the matrix space M_n^{sa} asymptotically approaches to the free analog when the matrix size goes to ∞. The following is our free TCI for probability measures on $\mathbf{R}$, where the relative entropy in the classical TCI is replaced by the relative free entropy (1.14).

Theorem 4.5. *Let Q be a real-valued function on $\mathbf{R}$. If $Q(x)-\frac{\rho}{2}x^2$ is convex on $\mathbf{R}$ with a constant $\rho>0$, then*

$$W(\mu,\mu_Q)\le\sqrt{\frac{1}{\rho}\widetilde{\Sigma}_Q(\mu)} \tag{4.3}$$

for every compactly supported $\mu\in\mathcal{M}(\mathbf{R})$.

In particular, when $Q(x) = x^2/2$ and so $\rho = 1$, the relative free entropy $\widetilde{\Sigma}_Q(\mu)$ is the inside of the square root in (4.2) and its minimizer is $\gamma_{0,2}$ so that Theorem 4.5 is a generalization of Theorem 4.4.

The next lemma will play a key role in our proof of the theorem.

Lemma 4.6. *Let $\tilde{\mu}, \tilde{\nu} \in \mathcal{M}(M_n^{sa})$ and $\hat{\mu}, \hat{\nu}$ be the mean eigenvalue distributions on $\mathbf{R}$ of $\tilde{\mu}, \tilde{\nu}$, respectively. Then*

$$W(\hat{\mu}, \hat{\nu}) \le \frac{1}{\sqrt{n}} W(\tilde{\mu}, \tilde{\nu}),$$

where $W(\tilde{\mu}, \tilde{\nu})$ is the Wasserstein distance with respect to the distance induced by the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ on M_n^{sa}.

Proof. For $A \in M_n^{sa}$ let $\lambda_1(A), \dots, \lambda_n(A)$ be the eigenvalues of A in increasing order with counting multiplicities. The mean eigenvalue distribution $\hat{\mu}$ is written as

$$\hat{\mu} = \int_{M_n^{sa}} \frac{1}{n} \left(\delta_{\lambda_1(A)} + \cdots + \delta_{\lambda_n(A)} \right) d\tilde{\mu}(A).$$

For each $\tilde{\pi} \in \Pi(\tilde{\mu}, \tilde{\nu})$ define $\hat{\pi} \in \mathcal{M}(\mathbf{R} \times \mathbf{R})$ by

$$\hat{\pi}(G) := \iint_{M_n^{sa} \times M_n^{sa}} \frac{1}{n} \#\{i : (\lambda_i(A), \lambda_i(B)) \in G\} \, d\tilde{\pi}(A, B)$$

for Borel sets $G \subset \mathbf{R} \times \mathbf{R}$. Since

$$\hat{\pi}(F \times \mathbf{R}) = \int_{M_n^{sa}} \frac{1}{n} \{\#\{i : \lambda_i(A) \in F\} \, d\tilde{\mu}(A) = \hat{\mu}(F)$$

and similarly $\hat{\pi}(\mathbf{R} \times F) = \hat{\nu}(F)$ for $F \subset \mathbf{R}$, we get $\hat{\pi} \in \Pi(\hat{\mu}, \hat{\nu})$ so that

$$\begin{aligned}
W(\hat{\mu}, \hat{\nu})^2 &\le \iint_{\mathbf{R} \times \mathbf{R}} \frac{1}{2} (x - y)^2 \, d\hat{\pi}(x, y) \\
&= \iint_{M_n^{sa} \times M_n^{sa}} \left\{ \iint_{\mathbf{R} \times \mathbf{R}} \frac{1}{2} (x - y)^2 \, d\left(\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i(A)} \otimes \delta_{\lambda_i(B)} \right) \right\} d\tilde{\pi}(A, B) \\
&= \frac{1}{n} \iint_{M_n^{sa} \times M_n^{sa}} \frac{1}{2} \sum_{i=1}^n \big(\lambda_i(A) - \lambda_i(B) \big)^2 \, d\tilde{\pi}(A, B).
\end{aligned}$$

The famous Lidskii-Wielandt majorization for Hermitian matrices (see [3]) implies that

$$\sum_{i=1}^n \big(\lambda_i(A) - \lambda_i(B) \big)^2 \le \sum_{i=1}^n \lambda_i(A - B)^2 = \|A - B\|_{HS}^2$$

for all $A, B \in M_n^{sa}$. Therefore,

$$W(\hat{\mu}, \hat{\nu})^2 \le \frac{1}{n} \iint_{M_n^{sa} \times M_n^{sa}} \frac{1}{2} \|A - B\|_{HS}^2 \, d\tilde{\pi}(A, B),$$

and taking the infimum over $\tilde{\pi} \in \Pi(\tilde{\mu}, \tilde{\nu})$ gives $W(\hat{\mu}, \hat{\nu})^2 \le \frac{1}{n} W(\tilde{\mu}, \tilde{\nu})^2$. □

Proof of Theorem 4.5. First, let $\mu \in \mathcal{M}(\mathbf{R})$ be compactly supported, and suppose that the function $Q_\mu(x) := 2 \int \log |x - y| \, d\mu(y)$ is finite and continuous on the whole $\mathbf{R}$. Choose $R > 0$ so that μ is supported in $[-R, R]$. For each $n \in \mathbf{N}$ consider the $n \times n$ self-adjoint random matrix $\lambda_n(Q_\mu; R) \in \mathcal{M}(M_n^{sa})$ supported in $\{A \in M_n^{sa} : \|A\|_\infty \le R\}$ as well as $\lambda_n(Q) \in \mathcal{M}(M_n^{sa})$ (see §§1.4 and §§1.5). Here, note that the condition (1.8) is automatically satisfied under the convexity assumption of Q. Since the corresponding large deviation principle

guarantees the weak convergence of the mean eigenvalue distribution $\hat\lambda_n(Q)$ (resp. $\hat\lambda_n(Q_\mu;R)$) to μ_Q (resp. μ), Lemma 4.1 gives

$$W(\mu,\mu_Q)\le\liminf_{n\to\infty}W\bigl(\hat\lambda_n(Q_\mu;R),\hat\lambda_n(Q)\bigr). \tag{4.4}$$

By Lemma 4.6 we get

$$W\bigl(\hat\lambda_n(Q_\mu;R),\hat\lambda_n(Q)\bigr)\le\frac{1}{\sqrt{n}}W\bigl(\lambda_n(Q_\mu;R),\lambda_n(Q)\bigr). \tag{4.5}$$

Set $\Psi_n(A):=n\mathrm{Tr}_n(Q(A))$ for $A\in M_n^{sa}$; then $d\lambda_n(Q)(A)=\frac{1}{Z_n(Q)}e^{-\Psi_n(A)}\,dA$. Since $Q(x)-\frac{\rho}{2}x^2$ is convex on $\mathbf{R}$, so is

$$\Psi_n(A)-\frac{\rho n}{2}\|A\|_{HS}^2=n\mathrm{Tr}_n\Bigl(Q(A)-\frac{\rho}{2}A^2\Bigr)\quad\text{on } M_n^{sa}.$$

Also, note that $\|\cdot\|_{HS}$ corresponds to the Euclidean norm on $\mathbf{R}^{n^2}$ under the isometry $A=[A_{ij}]\in M_n^{sa}\mapsto\bigl((A_{ii})_{1\le i\le n},\,(\sqrt{2}A_{ij})_{i<j}\bigr)\in\mathbf{R}^{n^2}$. Hence, Theorem 4.2 implies that

$$W\bigl(\lambda_n(Q_\mu;R),\lambda_n(Q)\bigr)\le\sqrt{\frac{1}{\rho n}S\bigl(\lambda_n(Q_\mu;R),\lambda_n(Q)\bigr)}. \tag{4.6}$$

Similarly to the case of special unitary random matrices in the proof of Theorem 3.3, since

$$\frac{d\lambda_n(Q_\mu;R)}{d\lambda_n(Q)}(A)=\frac{\widetilde Z_n(Q)}{\widetilde Z_n(Q_\mu;R)}\exp\bigl(-n\mathrm{Tr}_n(Q_\mu(A))+n\mathrm{Tr}_n(Q(A))\bigr)$$

on $(M_n^{sa})_R:=\{A\in M_n^{sa}:\|A\|_\infty\le R\}$, we have

$$\begin{aligned}
&\frac{1}{n^2}S\bigl(\lambda_n(Q_\mu;R),\lambda_n(Q)\bigr)\\
&\quad=\frac{1}{n^2}\log\widetilde Z_n(Q)-\frac{1}{n^2}\log\widetilde Z_n(Q_\mu)-\int_{(M_n^{sa})_R}\frac{1}{n}\mathrm{Tr}_n(Q_\mu(A))\,d\lambda_n(Q_\mu;R)(A)\\
&\qquad\qquad+\int_{(M_n^{sa})_R}\frac{1}{n}\mathrm{Tr}_n(Q(A))\,d\lambda_n(Q_\mu;R)(A)\\
&\quad\longrightarrow B(Q)-B(\mu;R)-\int_{[-R,R]}Q_\mu(x)\,d\mu(x)+\int_{\mathbf{R}}Q(x)\,d\mu(x)=\widetilde\Sigma_Q(\mu)
\end{aligned}\tag{4.7}$$

thanks to the fact that μ is the minimizer of the rate function (1.15) with Q_μ in place of Q. Combining (4.4)–(4.7) altogether implies the inequality (4.3) under the continuity assumption of $Q_\mu(x)$.

Finally, let $\mu\in\mathcal{M}(\mathbf{R})$ be a general compactly supported measure. By the regularization method in [17, p. 216] we can choose a sequence $\{\mu_k\}$ of measures in $\mathcal{M}(\mathbf{R})$ with compact supports uniformly bounded such that $Q_{\mu_k}(x)$ is continuous on $\mathbf{R}$ for each k, $\mu_k\to\mu$ weakly and $\Sigma(\mu_k)\ge\Sigma(\mu)$ for all k. Hence, by Lemma 4.1 and the first case we have

$$\begin{aligned}
W(\mu,\mu_Q)&\le\liminf_{n\to\infty}W(\mu_k,\mu_Q)\\
&\le\liminf_{k\to\infty}\sqrt{\frac{1}{\rho}\widetilde\Sigma_Q(\mu_k)}\le\sqrt{\frac{1}{\rho}\widetilde\Sigma_Q(\mu)},
\end{aligned}$$

completing the proof. ∎

5. Free TCI for measures on $\mathbf{T}$

In this section we will present the free analog of transportation cost inequalities for measures on $\mathbf{T}$. The idea with use of special unitary random matrices is the same as before. In the following we consider two kinds of Wasserstein distances between probability measures $\mu, \nu \in \mathcal{M}(\mathbf{T})$. The one is the Wasserstein distance with respect to the usual metric $|\zeta - \eta|$, $\zeta, \eta \in \mathbf{T}$, and the other is with respect to the geodesic distance (i.e., the angular distance) on $\mathbf{T}$. We write $W_{|\cdot|}(\mu, \nu)$ for the former and $W(\mu, \nu)$ for the latter. Of course, one has

$$W_{|\cdot|}(\mu, \nu) \le W(\mu, \nu), \qquad \mu, \nu \in \mathcal{M}(\mathbf{T}). \tag{5.1}$$

The next theorem is the free TCI for measures on $\mathbf{T}$ comparing the Wasserstein distance with the relative free entropy (1.17).

Theorem 5.1. *Let Q be a real-valued function on $\mathbf{T}$. If there exists a constant $\rho > -\frac{1}{2}$ such that $Q(e^{\sqrt{-1}t}) - \frac{\rho}{2}t^2$ is convex on $\mathbf{R}$, then*

$$W_{|\cdot|}(\mu, \mu_Q) \le W(\mu, \mu_Q) \le \sqrt{\frac{2}{1+2\rho}\widetilde{\Sigma}_Q(\mu)} \tag{5.2}$$

for every $\mu \in \mathcal{M}(\mathbf{T})$.

The special case where $Q \equiv 0$ and $\rho = 0$ is

$$W_{|\cdot|}\left(\mu, \frac{d\theta}{2\pi}\right) \le W\left(\mu, \frac{d\theta}{2\pi}\right) \le \sqrt{-2\Sigma(\mu)}, \qquad \mu \in \mathcal{M}(\mathbf{T}).$$

We need the next lemma to prove the theorem. Note that the lemma and the proof remain valid when $\mathrm{SU}(n)$ is replaced by $\mathrm{U}(n)$.

Lemma 5.2. *Let $\tilde\mu, \tilde\nu \in \mathcal{M}(\mathrm{SU}(n))$ and $W(\tilde\mu, \tilde\nu)$ be the Wasserstein distance between $\tilde\mu, \tilde\nu$ with respect to the geodesic distance on $\mathrm{SU}(n)$. Let $\hat\mu, \hat\nu$ be the mean eigenvalue distributions on $\mathbf{T}$ of $\tilde\mu, \tilde\nu$, respectively. Then*

$$W(\hat\mu, \hat\nu) \le \frac{1}{\sqrt{n}} W(\tilde\mu, \tilde\nu).$$

Proof. We use the symbol d for the geodesic distance on $\mathrm{SU}(n)$ as well as for that on $\mathbf{T}$. Define the optimal matching distance on $\mathbf{T}^n$ by

$$\delta(\zeta, \eta) := \min_{\sigma \in S_n} \sqrt{\sum_{i=1}^n d(\zeta_i, \eta_{\sigma(i)})^2}$$

for $\zeta = (\zeta_1, \dots, \zeta_n), \eta = (\eta_1, \dots, \eta_n) \in \mathbf{T}^n$. For $U \in \mathrm{SU}(n)$ let $\lambda(U) := (\lambda_1(U), \dots, \lambda_n(U))$ denote the element of $\mathbf{T}^n$ consisting of the eigenvalues of U with multiplicities and in counter-clockwise order (i.e., $0 \le \arg \lambda_1(U) \le \cdots \le \arg \lambda_n(U) < 2\pi$). First, we prove

$$\delta(\lambda(U), \lambda(V)) \le d(U, V), \qquad U, V \in \mathrm{SU}(n). \tag{5.3}$$

For $U, V \in \mathrm{SU}(n)$ let $U(t)$ $(0 \le t \le 1)$ be the geodesic curve in $\mathrm{SU}(n)$ connecting U and V. By dividing the curve into several small pieces if necessary, we may assume that there is a smooth curve $A(t)$ $(0 \le t \le 1)$ in $\{A \in M_n^{sa} : \mathrm{Tr}_n(A) = 0\}$ such that $U(t) = e^{\sqrt{-1}A(t)}$ for

$0 \le t \le 1$. Let $0 = t_0 < t_1 < \cdots < t_K = 1$ be any partition of $A(t)$. For $1 \le k \le K$ we have

$$\begin{aligned}\delta\bigl(\lambda(U(t_{k-1})), \lambda(U(t_k))\bigr) &\le \left\{\sum_{i=1}^n d\Bigl(e^{\sqrt{-1}\lambda_i(A(t_{k-1}))}, e^{\sqrt{-1}\lambda_i(A(t_k))}\Bigr)^2\right\}^{1/2} \\ &\le \left\{\sum_{i=1}^n |\lambda_i(A(t_{k-1})) - \lambda_i(A(t_k))|^2\right\}^{1/2} \\ &\le \|A(t_{k-1}) - A(t_k)\|_{HS} \\ &= d(U(t_{k-1}), U(t_k)) + o(t_k - t_{k-1}).\end{aligned}$$

In the above, $\lambda_1(A_k), \dots, \lambda_n(A_k)$ are the eigenvalues of A_k in increasing order, and the third inequality is due to the Lidskii-Wielandt majorization. Therefore,

$$\delta(\lambda(U), \lambda(V)) \le \sum_{k=1}^K \delta\bigl(\lambda(U(t_{k-1})), \lambda(U(t_k))\bigr) \le d(U, V) + o(1)$$

so that (5.3) follows because $o(1) \to 0$ as $\max_k(t_k - t_{k-1}) \to 0$.

Now, for each $U, V \in \mathrm{SU}(n)$ let $\sigma_{U,V} \in S_n$ be such that

$$\delta(\lambda(U), \lambda(V)) = \left\{\sum_{i=1}^n d\bigl(\lambda_i(U), \lambda_{\sigma_{U,V}(i)}(V)\bigr)^2\right\}^{1/2}.$$

Of course, we can let $(U, V) \in \mathrm{SU}(n) \times \mathrm{SU}(n) \mapsto \sigma_{U,V} \in S_n$ measurable. For every $\tilde\mu, \tilde\nu \in \mathcal{M}(\mathrm{SU}(n))$ and $\tilde\pi \in \Pi(\tilde\mu, \tilde\nu)$, define $\hat\pi \in \mathcal{M}(\mathbf{T} \times \mathbf{T})$ by

$$\hat\pi(G) := \iint_{\mathrm{SU}(n)\times\mathrm{SU}(n)} \frac{1}{n}\,\#\{i : (\lambda_i(U), \lambda_{\sigma_{U,V}(i)}(V)) \in G\}\, d\tilde\pi(U, V)$$

for Borel sets $G \subset \mathbf{T} \times \mathbf{T}$. Since for $F \subset \mathbf{T}$

$$\begin{aligned}\hat\pi(F \times \mathbf{T}) &= \int_{\mathrm{SU}(n)} \frac{1}{n}\,\#\{i : \lambda_i(U) \in F\}\, d\tilde\mu(U) = \hat\mu(F), \\ \hat\pi(\mathbf{T} \times F) &= \int_{\mathrm{SU}(n)} \frac{1}{n}\,\#\{i : \lambda_i(V) \in F\}\, d\tilde\nu(V) = \hat\nu(F),\end{aligned}$$

we have $\hat\pi \in \Pi(\hat\mu, \hat\nu)$ so that

$$\begin{aligned}W(\hat\mu, \hat\nu)^2 &\le \iint_{\mathbf{T}\times\mathbf{T}} \frac{1}{2} d(\zeta, \eta)^2\, d\hat\pi(\zeta, \eta) \\ &= \frac{1}{n}\iint_{\mathrm{SU}(n)\times\mathrm{SU}(n)} \frac{1}{2}\sum_{i=1}^n d\bigl(\lambda_i(U), \lambda_{\sigma_{U,V}(i)}(V)\bigr)^2\, d\tilde\pi(U, V) \\ &= \frac{1}{n}\iint_{\mathrm{SU}(n)\times\mathrm{SU}(n)} \frac{1}{2}\delta(\lambda(U), \lambda(V))^2\, d\tilde\pi(U, V) \\ &\le \frac{1}{n}\iint_{\mathrm{SU}(n)\times\mathrm{SU}(n)} \frac{1}{2} d(U, V)^2\, d\tilde\pi(U, V)\end{aligned}$$

thanks to (5.3). This implies $W(\hat\mu, \hat\nu)^2 \le \frac{1}{n} W(\tilde\mu, \tilde\nu)^2$. □

Proof of Theorem 5.1. The first inequality of (5.2) is obvious as noted in (5.1). To prove the second, we first assume:

(a) Q is harmonic on a neighborhood of the unit disk;
(b) the function $Q_\mu(\zeta) := 2\int_{\mathbf{T}} \log|\zeta - \eta|\, d\mu(\eta)$ is finite and continuous on $\mathbf{T}$.

For each $n \in \mathbf{N}$ define $\lambda_n^{\mathrm{SU}}(Q)$, $\lambda_n^{\mathrm{SU}}(Q_\mu)$ and $\hat\lambda_n^{\mathrm{SU}}(Q)$, $\hat\lambda_n^{\mathrm{SU}}(Q_\mu)$ as in the proof of Theorem 3.3. Since $\hat\lambda_n^{\mathrm{SU}}(Q) \to \mu_Q$ and $\hat\lambda_n^{\mathrm{SU}}(Q_\mu) \to \mu$ weakly, Lemma 4.1 implies that

$$W(\mu, \mu_Q) \le \liminf_{n\to\infty} W\bigl(\hat\lambda_n^{\mathrm{SU}}(Q_\mu), \hat\lambda_n^{\mathrm{SU}}(Q)\bigr). \tag{5.4}$$

On the other hand, Lemma 5.2 gives

$$W\bigl(\hat\lambda_n^{\mathrm{SU}}(Q_\mu), \hat\lambda_n^{\mathrm{SU}}(Q)\bigr) \le \frac{1}{\sqrt{n}} W\bigl(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\bigr). \tag{5.5}$$

Furthermore, since the function $\Psi_n(U) := n\mathrm{Tr}_n(Q(U))$ on $\mathrm{SU}(n)$ satisfies the Bakry and Emery criterion (3.2), Theorem 4.3 implies that

$$W\bigl(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\bigr) \le \sqrt{\frac{2}{n+2n\rho} S\bigl(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\bigr)}. \tag{5.6}$$

The above (5.4)–(5.6) and (3.5) (see also Proposition 6.1 (1) in §6) altogether prove the second inequality of (5.2) under assumptions (a) and (b).

Next, let Q be as stated in the theorem (hence Q is continuous on $\mathbf{T}$) and $\mu \in \mathcal{M}(\mathbf{T})$ be general. For $0 < r < 1$ let the Poisson integrals Q_r, p_r and μ_r be as in the proof of Theorem 3.3. Since Q_r and μ_r satisfy (a) and (b) above, the case already shown implies that

$$W(\mu_r, \mu_{Q_r}) \le \sqrt{\frac{2}{1+2\rho} \widetilde\Sigma_{Q_r}(\mu_r)}. \tag{5.7}$$

Moreover, as in the proof of Theorem 3.3, we have $\|Q_r - Q\| \to 0$, $B(Q_r) \to B(Q)$ and $\widetilde\Sigma_{Q_r}(\mu_r) \to \widetilde\Sigma_Q(\mu)$ as $r \nearrow 1$. Choose any sequence $0 < r(k) < 1$ with $r(k) \to 1$ such that $\mu_{Q_{r(k)}} \to \mu_0 \in \mathcal{M}(\mathbf{T})$ weakly. By the upper semicontinuity of $\Sigma(\mu)$, we get

$$0 \le \widetilde\Sigma_Q(\mu_0) \le \liminf_{k\to\infty} \widetilde\Sigma_{Q_{r(k)}}(\mu_{Q_{r(k)}}) = 0$$

so that $\mu_0 = \mu_Q$. This shows that $\mu_{Q_r} \to \mu_Q$ weakly as $r \nearrow 1$ and

$$W(\mu, \mu_Q) \le \liminf_{r\nearrow 1} W(\mu_r, \mu_{Q_r})$$

thanks to Lemma 4.1. Hence, the desired inequality finally follows by taking the limit of (5.7). □

6. Concluding remarks

In this section we collect some remarks, examples and supplementary results.

6.1. **Use of special orthogonal random matrices.** For a real-valued continuous function Q, an $n \times n$ special orthogonal random matrix $\lambda_n^{\mathrm{SO}}(Q)$ is defined by

$$d\lambda_n^{\mathrm{SO}}(Q)(V) := \frac{1}{Z_n^{\mathrm{SO}}(Q)} \exp\Bigl(-\frac{n}{2}\mathrm{Tr}_n(Q(V))\Bigr)\, dV,$$

where dV is the Haar probability measure on the special orthogonal group $\mathrm{SO}(n)$. The joint eigenvalue distribution on $\mathbf{T}^{n-1}$ of $\lambda_n^{\mathrm{SO}}(Q)$ is

$$d\tilde\lambda_n^{\mathrm{SO}}(Q)(\zeta_1, \dots, \zeta_{n-1}) = \frac{1}{\widetilde Z_n^{\mathrm{SO}}(Q)} \exp\left(-\frac{n}{2}\sum_{i=1}^n Q(\zeta_i)\right) \prod_{1\le i<j\le n} |\zeta_i - \zeta_j| \prod_{i=1}^n d\zeta_i$$
$$\text{with} \quad \zeta_n = (\zeta_1 \cdots \zeta_{n-1})^{-1}.$$

The large deviation is analogous to Theorem 1.2; the rate function is just $\frac{1}{2}\widetilde{\Sigma}_Q(\mu)$ and its minimizer is the same μ_Q. On the other hand, note that the Ricci curvature tensor of $\mathrm{SO}(n)$ is

$$\mathrm{Ric}(\mathrm{SO}(n)) = \frac{n-2}{4} I_{n(n-1)/2},$$

and the Bakry and Emery criterion in place of (3.2) is

$$\mathrm{Ric}(\mathrm{SO}(n)) + \mathrm{Hess}(\Psi_n) \ge \left(\frac{n-2}{4} + \frac{n}{2}\rho\right) I_{n(n-1)/2},$$

where $\Psi_n(V) := \frac{n}{2}\mathrm{Tr}_n(Q(V))$ for $V \in \mathrm{SO}(n)$. In this way, a special orthogonal random matrix model can be used as well to obtain the free LSI in Theorem 3.3 and the free TCI in Theorem 5.1. Similarly, the free TCI in Theorem 4.5 can be shown by using a real symmetric random matrix model

$$d\lambda_n^{\mathrm{real}}(Q)(T) := \frac{1}{Z_n^{\mathrm{real}}(Q)} \exp\Bigl(-\frac{n}{2}\mathrm{Tr}_n(Q(T))\Bigr)\, dT,$$

where $dT := \prod_{i\le j} dT_{ij}$ on $M_n(\mathbf{R})^{sa} \cong \mathbf{R}^{n(n+1)/2}$.

6.2. **Some computations.** Let $Q(x) := \rho x^2/2$ on $\mathbf{R}$ with $\rho > 0$. The equilibrium measure associated with Q is the semicircular measure $\gamma_{0,2/\sqrt{\rho}}$. For $\alpha > 0$ we compute

$$\begin{aligned}
\widetilde{\Sigma}_Q(\gamma_{0,2/\sqrt{\alpha}}) &= \frac{1}{2}\log\alpha + \frac{\rho}{2\alpha} - \frac{1}{2}\log\rho - \frac{1}{2}, \\
\Phi_Q(\gamma_{0,2/\sqrt{\alpha}}) &= \frac{(\alpha-\rho)^2}{\alpha}.
\end{aligned}$$

Since

$$\lim_{\alpha\to\rho} \frac{\widetilde{\Sigma}_Q(\gamma_{0,2/\sqrt{\alpha}})}{\Phi_Q(\gamma_{0,2/\sqrt{\alpha}})} = \frac{1}{4\rho},$$

we notice that the bound $1/2\rho$ in the free LSI (2.2) cannot be smaller than $1/4\rho$; however it is unknown whether $1/2\rho$ is the best possible bound or not.

For $2 \le \lambda \le \infty$ the equilibrium measure associated with $Q(\zeta) := -(2/\lambda)\mathrm{Re}\,\zeta$ on $\mathbf{T}$ is

$$\nu_\lambda := \left(1 + \frac{2}{\lambda}\cos\theta\right)\frac{d\theta}{2\pi} \qquad \left(\text{with } \nu_\infty = \frac{d\theta}{2\pi}\right) \tag{6.1}$$

and $\Sigma(\nu_\lambda) = -1/\lambda^2$ (see [17, 5.3.10]). When $4 < \lambda \le \infty$, since $Q\left(e^{\sqrt{-1}t}\right) + \frac{1}{\lambda}t^2 = \frac{2}{\lambda}\left(\frac{t^2}{2} - \cos t\right)$ is convex on $\mathbf{R}$, the free LSI (3.1) holds with $1/(1+2\rho) = \lambda/(\lambda-4)$. For example, for $2 \le \alpha \le \infty$ we compute

$$\widetilde{\Sigma}_Q(\nu_\alpha) = \Bigl(\frac{1}{\alpha} - \frac{1}{\lambda}\Bigr)^2, \quad F_Q(\nu_\alpha) = 2\Bigl(\frac{1}{\alpha} - \frac{1}{\lambda}\Bigr)^2.$$

Again, the optimality of the bound $1/(1+2\rho)$ in (3.1) is unknown.

Concerning the free TCI, it does not seem easy to exactly compute the Wasserstein distance; in fact, we do not know the exact value of $W(\gamma_{0,r_1}, \gamma_{0,r_2})$ for instance.

6.3. **Classical TCI vs. free TCI.** Both classical and free TCI's are formulated in terms of the same (quadratic) Wasserstein distance for measures, and thus it seems interesting to compare these two. However, in the case of measures on $\mathbf{R}$, the natural reference measures are Gaussian (not being compactly supported) in the classical case, while semicircular (being compactly supported) in the free case, and hence the question is irrelevant in this case. In the case of the uniform probability measure $d\theta/2\pi$ on $\mathbf{T}$, our free TCI is

$$W\left(\mu, \frac{d\theta}{2\pi}\right) \le \sqrt{-2\Sigma(\mu)}, \qquad \mu \in \mathcal{M}(\mathbf{T}),$$

while to the authors' best knowledge the sharpest classical TCI is

$$W\left(\mu, \frac{d\theta}{2\pi}\right) \le \sqrt{S\left(\mu, \frac{d\theta}{2\pi}\right)}, \qquad \mu \in \mathcal{M}(\mathbf{T}).$$

(The latter inequality is seen as follows. It is known (see [21, p.94]) that the "spectral gap" and "logarithmic Sobolev constant" are the same number 1, and [25, Theorem 1] implies the desired inequality.) Now, if the relative free entropy happens to dominate the (usual) relative entropy up to a positive constant, then a free TCI would immediately follow from the classical one. However, this is not, and we indeed have the following examples:

(1) For an arbitrary $k \in \mathbf{N}$ and for large $n \in \mathbf{N}$, let us choose k disjoint intervals $[a_j(n), b_j(n)]$, $1 \le j \le k$, in $\mathbf{T} = [0, 2\pi)$ whose lengths are all $2\pi/kn$ and whose center points are fixed independently of the choice n. Consider $\mu_k(n) \in \mathcal{M}(\mathbf{T})$ whose density is $\sum_{j=1}^k n\chi_{[a_j(n),b_j(n)]}$. Then we have

$$S\left(\mu_k(n), \frac{d\theta}{2\pi}\right) = \log n.$$

On the other hand, by a straightforward computation we see that, for a sufficiently large $n_0 \in \mathbf{N}$, there are constants $c_k < C_k$ depending only on k such that

$$c_k + \frac{\log n}{k} \le -\Sigma\left(\mu_k(n)\right) \le C_k + \frac{\log n}{k} \quad \text{for } n \ge n_0,$$

and thus

$$\frac{-\Sigma\left(\mu_k(n)\right)}{S\left(\mu_k(n), \frac{d\theta}{2\pi}\right)} \to \frac{1}{k} \quad \text{as } n \to \infty.$$

The computation is somewhat similar to a free entropy dimension computation for single variables; see [27, Proposition 6.1] for example.

(2) For the measure ν_λ $(2 < \lambda < \infty)$ in (6.1), with the help of a table on integration formulas, we can compute

$$S\left(\nu_\lambda, \frac{d\theta}{2\pi}\right) = \log\left(\frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{\lambda^2}}\right)\right) + 1 + \frac{4}{\lambda\sqrt{\lambda^2 - 4}} - \frac{1}{\sqrt{1 - \frac{4}{\lambda^2}}},$$

and hence we get

$$\frac{S\left(\nu_\lambda, \frac{d\theta}{2\pi}\right)}{-\Sigma\left(\nu_\lambda\right)} \longrightarrow 0 \quad \text{as } \lambda \to \infty.$$

These examples tell us that the minus free entropy $-\Sigma(\mu)$ cannot be compared with the relative entropy $S(\mu, d\theta/2\pi)$.

6.4. **Scaling limit formulas for relative free entropy and relative free Fisher information.** It seems worthwhile to state some scaling limit formulas given in the proofs of the main theorems in separate propositions, saying that the relative entropy and the Fisher information of relevant random matrices asymptotically converge to the corresponding free analogs for limiting measures. In fact, the formulas for relative free entropy were essentially got in [13].

The proof of (3.5) gives (1) of the next proposition, while that of (3.6) does (2) because Lemma 1.3 shows that the derivative formula in Lemma 3.1 (ii) is still valid for any $U \in \mathrm{SU}(n)$ when Q is a real-valued C^1 function on $\mathbf{T}$. The unitary versions are similar.

Proposition 6.1. (1) *Let Q be a real-valued continuous function on $\mathbf{T}$, and $\mu \in \mathcal{M}(\mathbf{T})$. If $Q_\mu(\zeta) := 2\int_{\mathbf{T}} \log|\zeta - \eta|\,d\mu(\eta)$ is finite and continuous on $\mathbf{T}$, then*

$$\widetilde{\Sigma}_Q(\mu) = \lim_{n\to\infty} \frac{1}{n^2} S\bigl(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\bigr) = \lim_{n\to\infty} \frac{1}{n^2} S\bigl(\lambda_n^{\mathrm{U}}(Q_\mu), \lambda_n^{\mathrm{U}}(Q)\bigr).$$

(2) *In addition, if μ has a continuous density $d\mu/d\zeta$ and both Q and Q_μ are C^1 functions on $\mathbf{T}$, then*

$$\begin{aligned} F_Q(\mu) &= \lim_{n\to\infty} \frac{1}{n^3} \int_{\mathrm{SU}(n)} \left\| \nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U) \right\|_{HS}^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\ &= \lim_{n\to\infty} \frac{1}{n^3} \int_{\mathrm{U}(n)} \left\| \nabla \log \frac{d\lambda_n^{\mathrm{U}}(Q_\mu)}{d\lambda_n^{\mathrm{U}}(Q)}(U) \right\|_{HS}^2 d\lambda_n^{\mathrm{U}}(Q_\mu)(U). \end{aligned}$$

Similar limit formulas are given also in the real line case. The formula in (1) below is (4.7). The proof of (2) is more or less similar to the circle case; here the fact that $Q_\mu'(x) = 2(Hp)(x)$ for a.e. $x \in \mathbf{R}$ is needed in place of Lemma 3.2 (i). The details are left to the reader. Note that the limits in both formulas are independent of the choice of R such that μ is supported in $[-R, R]$. Although the assumption of Q_μ being C^1 on $\mathbf{R}$ seems rather strong, yet we have many such examples (see [26, §IV.5]).

Proposition 6.2. (1) *Let Q be a real-valued continuous function on $\mathbf{R}$ satisfying* (1.8)*, and $\mu \in \mathcal{M}(\mathbf{R})$ be supported in $[-R, R]$. If $Q_\mu(x) := 2\int_{\mathbf{R}} \log|x - y|\,d\mu(x)$ is finite and continuous on $\mathbf{R}$, then*

$$\widetilde{\Sigma}_Q(\mu) = \lim_{n\to\infty} \frac{1}{n^2} S\bigl(\lambda_n(Q_\mu; R), \lambda_n(Q)\bigr).$$

(2) *In addition, if μ has a continuous density $d\mu/dx$ and both Q and Q_μ are C^1 functions on $\mathbf{R}$, then*

$$\Phi_Q(\mu) = \lim_{n\to\infty} \frac{1}{n^3} \int_{(M_n^{sa})_R} \left\| \nabla \log \frac{d\lambda_n(Q_\mu; R)}{d\lambda_n(Q)}(A) \right\|_{HS}^2 d\lambda_n(Q_\mu; R)(A).$$

6.5. **Free LSI for measures on $\mathbf{R}^+$.** The free LSI (2.2) is applicable in particular for measures supported in $\mathbf{R}^+ = [0, \infty)$, but we can also show a different inequality which might be a proper free LSI in the case where the whole space is $\mathbf{R}^+$ instead of $\mathbf{R}$. Let $\mathcal{M}_s(\mathbf{R})$ be the set of symmetric probability measures on $\mathbf{R}$. Consider the bijective transformation $\mu \in \mathcal{M}(\mathbf{R}^+) \mapsto \tilde{\mu} \in \mathcal{M}_s(\mathbf{R})$ defined as

$$\mu(F) = \tilde{\mu}(\{x \in \mathbf{R} : x^2 \in F\}) \quad \text{for } F \subset \mathbf{R}^+.$$

When $\mu \in \mathcal{M}(\mathbf{R}^+)$ has the density $p = d\mu/dx$ on $\mathbf{R}^+$, the measure $\tilde{\mu}$ has the density $\tilde{p} = d\tilde{\mu}/dx$ on $\mathbf{R}$ and

$$\tilde{p}(x) = |x|p(x^2), \quad x \in \mathbf{R};$$
$$p(x) = \frac{\tilde{p}(\sqrt{x})}{\sqrt{x}}, \quad x \in \mathbf{R}^+.$$

Lemma 6.3. *Let f be a measurable function on $\mathbf{R}^+$ and set $\tilde{f}(x) := |x|f(x^2)$ for $x \in \mathbf{R}$. Then $\tilde{f} \in L^3(\mathbf{R}, dx)$ if and only $f \in L^3(\mathbf{R}^+, x\,dx)$. If this is the case, then the Hilbert transform $(Hf)(x)$ exists for a.e. $x \in \mathbf{R}^+$ and $(H\tilde{f})(x) = x(Hf)(x^2)$ for a.e. $x \in \mathbf{R}$.*

Proof. The first assertion is seen because $\int_{\mathbf{R}} |\tilde{f}(x)|^3\,dx = \int_{\mathbf{R}^+} x|f(x)|^3\,dx$. Suppose $f \in L^3(\mathbf{R}^+, x\,dx)$; then $(H\tilde{f})(x)$ exists for a.e. $x \in \mathbf{R}$. For every $x > 0$ and $0 < \varepsilon < x^2$ we compute

$$\begin{aligned}
&x\left(\int_0^{x^2-\varepsilon} + \int_{x^2+\varepsilon}^{\infty}\right) \frac{f(t)}{x^2 - t}\,dt \\
&= \left(\int_0^{x^2-\varepsilon} + \int_{x^2+\varepsilon}^{\infty}\right)\left(\frac{1}{x+\sqrt{t}} + \frac{1}{x-\sqrt{t}}\right)\frac{\tilde{f}(\sqrt{t})}{2\sqrt{t}}\,dt \\
&= \left(\int_0^{\sqrt{x^2-\varepsilon}} + \int_{\sqrt{x^2+\varepsilon}}^{\infty}\right)\left(\frac{1}{x+s} + \frac{1}{x-s}\right)\tilde{f}(s)\,ds \\
&= \left(\int_{-\infty}^{\sqrt{x^2-\varepsilon}} + \int_{2x-\sqrt{x^2-\varepsilon}}^{\infty}\right)\frac{\tilde{f}(s)}{x-s}\,ds - \int_{-\sqrt{x^2+\varepsilon}}^{-\sqrt{x^2-\varepsilon}} \frac{\tilde{f}(s)}{x-s}\,ds + \int_{\sqrt{x^2+\varepsilon}}^{2x-\sqrt{x^2-\varepsilon}} \frac{\tilde{f}(s)}{x-s}\,ds.
\end{aligned}$$

The first of the last three terms is the principal value integral converging to $(H\tilde{f})(x)$ as $\varepsilon \searrow 0$ for a.e. $x > 0$, while the second and the third terms converge to 0 as $\varepsilon \searrow 0$. Indeed,

$$\int_{-\sqrt{x^2+\varepsilon}}^{-\sqrt{x^2-\varepsilon}} \left|\frac{\tilde{f}(s)}{x-s}\right| ds \le \frac{1}{x+\sqrt{x^2-\varepsilon}} \int_{-\sqrt{x^2+\varepsilon}}^{-\sqrt{x^2-\varepsilon}} |\tilde{f}(s)|\,ds \longrightarrow 0$$

and

$$\begin{aligned}
&\int_{\sqrt{x^2+\varepsilon}}^{2x-\sqrt{x^2-\varepsilon}} \left|\frac{\tilde{f}(s)}{x-s}\right| ds \\
&\quad\le \left(\int_{-\infty}^{\infty} \left|\tilde{f}(s)\right|^3 ds\right)^{1/3} \left(\int_{\sqrt{x^2+\varepsilon}}^{2x-\sqrt{x^2-\varepsilon}} \frac{ds}{(s-x)^{3/2}}\right)^{2/3} \\
&\quad= \left(\int_{-\infty}^{\infty} \left|\tilde{f}(s)\right|^3 ds\right)^{1/3} \frac{2}{\sqrt{\varepsilon}}\left(\left(\sqrt{x^2+\varepsilon}+x\right)^{1/2} - \left(x+\sqrt{x^2-\varepsilon}\right)^{1/2}\right) \\
&\quad\longrightarrow 0 \quad \text{as } \varepsilon \searrow 0.
\end{aligned}$$

Therefore, we see that $(Hf)(x^2)$ exists and $(H\tilde{f})(x) = x(Hf)(x^2)$ for a.e. $x > 0$. Moreover, we have $(H\tilde{f})(x) = -(H\tilde{f})(-x) = x(Hf)(x^2)$ for a.e. $x < 0$ as well. ∎

Let Q be a real-valued C^1 function on $\mathbf{R}^+$. For each $\mu \in \mathcal{M}(\mathbf{R}^+)$ we define the "relative free Fisher information" $\Phi_Q^+(\mu)$ to be

$$\Phi_Q^+(\mu) := 4\int_{\mathbf{R}^+} x\left((Hp)(x) - \frac{1}{2}Q'(x)\right)^2 d\mu(x)$$

when μ has the density $p = d\mu/dx$ belonging to $L^3(\mathbf{R}^+, x\,dx)$; otherwise to be $+\infty$. In particular, the "free Fisher information" $\Phi^+(\mu)$ is defined as $\Phi_Q^+(\mu)$ with $Q \equiv 0$, i.e.,

$$\Phi^+(\mu) = 4\int_{\mathbf{R}^+} x(Hp(x))^2\,d\mu(x).$$

On the other hand, let Q be a real-valued continuous function on $\mathbf{R}^+$ such that

$$\lim_{x\to+\infty} x\exp(-\varepsilon Q(x)) = 0 \quad \text{for any } \varepsilon > 0.$$

We define the "relative free entropy" $\widetilde{\Sigma}_Q^+(\mu)$ of $\mu \in \mathcal{M}(\mathbf{R}^+)$ as

$$\widetilde{\Sigma}_Q^+(\mu) := -\Sigma(\mu) + \int_{\mathbf{R}^+} Q(x)\,d\mu(x) + B^+(Q),$$

where

$$B^+(Q) := \lim_{n\to\infty}\frac{1}{n^2}\int\cdots\int_{(\mathbf{R}^+)^n}\exp\left(-n\sum_{i=1}^n Q(x_i)\right)\prod_{i<j}(x_i - x_j)^2\prod_{i=1}^n dx_i.$$

In fact, similarly to the real line case in §§1.4, the function $\widetilde{\Sigma}_Q^+(\mu)$ on $\mathcal{M}(\mathbf{R}^+)$ is the good rate function of the large deviation principle for the empirical eigenvalue distribution of the $n\times n$ positive random matrix

$$d\lambda_n^+(Q)(A) := \frac{1}{Z_n^+(Q)}\exp\bigl(-n\mathrm{Tr}_n(Q(A))\bigr)\chi_{\{A\ge 0\}}(A)\,dA.$$

Proposition 6.4. *Let Q be a real-valued convex continuous function on $\mathbf{R}^+$ such that Q is C^1 on $(0,\infty)$ and $Q'(x) \ge \rho$ for all $x > 0$ with a constant $\rho > 0$. Then, for every $\mu \in \mathcal{M}(\mathbf{R}^+)$ one has*

$$\widetilde{\Sigma}_Q^+(\mu) \le \frac{1}{\rho}\Phi_Q^+(\mu). \tag{6.2}$$

Proof. Define $\widetilde{Q}(x) := \frac{1}{2}Q(x^2)$ for $x \in \mathbf{R}$; then it is easy to check that $\widetilde{Q}$ is a C^1-function on $\mathbf{R}$ and $Q(x) - \frac{\rho}{2}x^2$ is convex on $\mathbf{R}$. For each $\mu \in \mathcal{M}(\mathbf{R}^+)$ we can apply Theorem 2.2 to $\tilde{\mu} \in \mathcal{M}_s(\mathbf{R})$ defined as above so that

$$\widetilde{\Sigma}_{\widetilde{Q}}(\tilde{\mu}) \le \frac{1}{2\rho}\Phi_{\widetilde{Q}}(\tilde{\mu}).$$

Now, it suffices to show that

$$\Phi_Q^+(\mu) = \Phi_{\widetilde{Q}}(\tilde{\mu}), \tag{6.3}$$

$$\widetilde{\Sigma}_Q^+(\mu) = 2\widetilde{\Sigma}_{\widetilde{Q}}(\tilde{\mu}). \tag{6.4}$$

To prove (6.3), we may assume that μ has the density $p = d\mu/dx \in L^3(\mathbf{R}^+, x\,dx)$. Letting $\tilde{p} = d\tilde{\mu}/dx \in L^3(\mathbf{R}, dx)$, we get by Lemma 6.3

$$\begin{aligned}\Phi_Q^+(\mu) &= 4\int_{\mathbf{R}^+} x\left((Hp)(x) - \frac{1}{2}Q'(x)\right)^2 p(x)\,dx\\ &= 8\int_{\mathbf{R}^+}\left((H\tilde{p})(x) - \frac{1}{2}\widetilde{Q}'(x)\right)^2\tilde{p}(x)\,dx = \Phi_{\widetilde{Q}}(\tilde{\mu}).\end{aligned}$$

For every $\mu \in \mathcal{M}(\mathbf{R}^+)$ we have $\Sigma(\mu) = 2\Sigma(\tilde{\mu})$ (see [17, p. 198]) and $\int_{\mathbf{R}^+} Q(x)\,d\mu(x) = 2\int_{\mathbf{R}}\widetilde{Q}(x)\,d\tilde{\mu}(x)$. For each $\nu \in \mathcal{M}(\mathbf{R})$, setting $\nu' \in \mathcal{M}(\mathbf{R})$ by $\nu'(F) := \nu(-F)$, we get $\widetilde{\Sigma}_{\widetilde{Q}}((\nu+\nu')/2) \le \widetilde{\Sigma}_{\widetilde{Q}}(\nu)$ by the concavity of free entropy (see [17, p. 193]). These facts show

that the equilibrium measure $\mu_{\widetilde{Q}}$ associated with $\widetilde{Q}$ coincides with $\tilde{\mu}_Q$ where μ_Q is the unique minimizer of $\widetilde{\Sigma}_Q^+(\mu)$. Therefore, we see that $B^+(Q) = 2B(\widetilde{Q})$ and $\widetilde{\Sigma}_Q^+(\mu) = 2\widetilde{\Sigma}_{\widetilde{Q}}(\tilde{\mu})$ for all $\mu \in \mathcal{M}(\mathbf{R}^+)$. □

In particular, when $Q(x) = \rho x$ on $\mathbf{R}^+$ with $\rho > 0$, note that $\tilde{\mu}_Q = \gamma_{0,2/\sqrt{\rho}}$ for the unique minimizer μ_Q of $\Sigma_Q^+(\mu)$, and the inequality (6.2) becomes

$$-\Sigma(\mu) + \rho \int_{\mathbf{R}^+} x\,d\mu(x) - \log\rho - \frac{3}{2} \le \frac{1}{\rho}\left(\Phi^+(\mu) - 2\rho + \rho^2 \int_{\mathbf{R}^+} x\,d\mu(x)\right),$$

that is,

$$\chi(\mu) \ge -\frac{1}{\rho}\Phi^+(\mu) - \log\rho + \frac{1}{2}\log 2\pi + \frac{5}{4}$$

as long as $\int_{\mathbf{R}^+} x\,d\mu(x) < +\infty$. Maximizing the above right-hand side over $\rho > 0$ gives

$$\chi(\mu) \ge \frac{1}{2}\log\frac{2\pi e^{1/2}}{\Phi^+(\mu)^2}, \tag{6.5}$$

which also follows from (2.4) combined with $\Sigma(\mu) = 2\Sigma(\tilde{\mu})$ and $\Phi^+(\mu) = \Phi(\tilde{\mu})$. Notice that $\Phi(\mu)$ in (2.4) and $\Phi^+(\mu)^2$ in (6.5) are not comparable. For example, when $\mu \in \mathcal{M}(\mathbf{R}^+)$ has a density $p(x) = (\alpha+1)x^\alpha\chi_{(0,1]}(x)$ with $\alpha > -1/3$, we compute

$$\Phi(\mu) = \frac{4(\alpha+1)^3}{3(3\alpha+1)}, \quad \Phi^+(\mu) = \frac{4(\alpha+1)^3}{3(3\alpha+2)},$$

so that $\Phi^+(\mu)^2/\Phi(\mu)$ converges to 0 as $\alpha \to -1/3$ and also to $+\infty$ as $\alpha \to +\infty$.

6.6. **Free TCI for measures on $\mathbf{R}^+$.** Consider the bijective transformation $\mu \in \mathcal{M}(\mathbf{R}^+) \mapsto \hat{\mu} \in \mathcal{M}(\mathbf{R}^+)$ defined as

$$\mu(F) = \hat{\mu}(\{x \in \mathbf{R}^+ : x^2 \in F\}) \quad \text{for } F \subset \mathbf{R}^+.$$

The next proposition is a free TCI when the whole space is $\mathbf{R}^+$.

Proposition 6.5. *Let Q be a real-valued function on $\mathbf{R}^+$. If $Q(x^2) - \rho x^2$ is convex on $\mathbf{R}$ with a constant $\rho > 0$, then*

$$W(\hat{\mu}, \hat{\mu}_Q) \le \sqrt{\frac{1}{2\rho}\widetilde{\Sigma}_Q^+(\mu)}$$

for every compactly supported $\mu \in \mathcal{M}(\mathbf{R}^+)$, where μ_Q is the minimizer of $\widetilde{\Sigma}_Q^+(\mu)$.

Proof. Let $\widetilde{Q}(x) := \frac{1}{2}Q(x^2)$ for $x \in \mathbf{R}$ as in the proof of Proposition 6.4. Since $\tilde{\mu}_Q$ is the minimizer of $\widetilde{\Sigma}_{\widetilde{Q}}(\nu)$ for $\nu \in \mathcal{M}(\mathbf{R})$, Theorem 4.5 and (6.4) imply that

$$W(\tilde{\mu}, \tilde{\mu}_Q) \le \sqrt{\frac{1}{\rho}\widetilde{\Sigma}_{\widetilde{Q}}(\tilde{\mu})} = \sqrt{\frac{1}{2\rho}\widetilde{\Sigma}_Q^+(\mu)}.$$

Hence, it remains to show that

$$W(\hat{\mu}, \hat{\mu}_Q) \le W(\tilde{\mu}, \tilde{\mu}_Q).$$

To prove this, let $\pi \in \Pi(\tilde{\mu}, \tilde{\mu}_Q)$ and define

$$\hat{\pi}(G) := \pi\big(\{(x,y) \in \mathbf{R} \times \mathbf{R} : (|x|, |y|) \in G\}\big)$$

for Borel sets $G \subset \mathbf{R}^+ \times \mathbf{R}^+$. Then we get $\hat\pi \in \Pi(\hat\mu, \hat\mu_Q)$ so that

$$\begin{aligned} W(\hat\mu, \hat\mu_Q) &\le \int_{\mathbf{R}^+\times\mathbf{R}^+} \frac{1}{2}(x-y)^2\,d\hat\pi(x,y) = \int_{\mathbf{R}\times\mathbf{R}} \frac{1}{2}(|x|-|y|)^2\,d\pi(x,y) \\ &\le \int_{\mathbf{R}\times\mathbf{R}} \frac{1}{2}(x-y)^2\,d\pi(x,y). \end{aligned}$$

This implies the desired inequality. □

By replacing $\hat\mu$ by μ, the above free TCI can be rewritten as

$$\begin{aligned} &W(\mu, \hat\mu_Q) \\ &\le \sqrt{\frac{1}{2\rho}\left(-\iint_{\mathbf{R}^+\times\mathbf{R}^+} \log|x^2-y^2|\,d\mu(x)\,d\mu(y) + \int_{\mathbf{R}^+} Q(x^2)\,d\mu(x) + B^+(Q)\right)} \end{aligned}$$

for every compactly supported $\mu \in \mathcal{M}(\mathbf{R}^+)$. For example, when $Q(x) = x$ on $\mathbf{R}^+$ and $\rho = 1$, $\hat\mu_Q$ is the quarter-semicircular distribution $\frac{1}{\pi}\sqrt{4-x^2}\,\chi_{[0,2]}\,dx$ and $B^+(Q) = -3/2$.

Appendix A. Proof of Theorem 1.2

In the following let us keep the relation $\zeta_n = (\zeta_1\cdots\zeta_{n-1})^{-1}$. The proof below is essentially same as that in [16]. Set

$$F(\zeta,\eta) := -\log|\zeta-\eta| + \frac{1}{2}(Q(\zeta)+Q(\eta)).$$

As in [16] it suffices to prove the following inequalities:

(i)
$$\limsup_{n\to\infty} \frac{1}{n^2}\log \widetilde{Z}_n^{\mathrm{SU}}(Q) \le -\inf_{\mu\in\mathcal{M}(\mathbf{T})} \iint_{\mathbf{T}^2} F(\zeta,\eta)\,d\mu(\zeta)\,d\mu(\eta).$$

(ii) For every $\mu \in \mathcal{M}(\mathbf{T})$,
$$\begin{aligned} &\inf_G \left[\limsup_{n\to\infty} \frac{1}{n^2}\log \tilde\lambda_n^{\mathrm{SU}}(Q)\left\{\frac{1}{n}\bigl(\delta_{\zeta_1}+\cdots+\delta_{\zeta_{n-1}}+\delta_{\zeta_n}\bigr)\in G\right\}\right] \\ &\qquad \le -\iint_{\mathbf{T}^2} F(\zeta,\eta)\,d\mu(\zeta)\,d\mu(\eta) - \liminf_{n\to\infty}\frac{1}{n^2}\log\widetilde{Z}_n^{\mathrm{SU}}(Q), \end{aligned}$$
where G runs over all neighborhoods of μ.

(iii) For every $\mu \in \mathcal{M}(\mathbf{T})$,
$$\liminf_{n\to\infty}\frac{1}{n^2}\log\widetilde{Z}_n^{\mathrm{SU}}(Q) \ge -\iint_{\mathbf{T}^2} F(\zeta,\eta)\,d\mu(\zeta)\,d\mu(\eta).$$

(iv) For every $\mu \in \mathcal{M}(\mathbf{T})$,
$$\begin{aligned} &\inf_G \left[\liminf_{n\to\infty} \frac{1}{n^2}\log \tilde\lambda_n^{\mathrm{SU}}(Q)\left\{\frac{1}{n}\bigl(\delta_{\zeta_1}+\cdots+\delta_{\zeta_{n-1}}+\delta_{\zeta_n}\bigr)\in G\right\}\right] \\ &\qquad \ge -\iint_{\mathbf{T}^2} F(\zeta,\eta)\,d\mu(\zeta)\,d\mu(\eta) - \limsup_{n\to\infty}\frac{1}{n^2}\log\widetilde{Z}_n^{\mathrm{SU}}(Q), \end{aligned}$$
where G is as in (ii).

The proofs of the first two are the same as in [16], so we omit them. To prove (iii) and (iv), we may assume (see [16]) that μ has a continuous density $f > 0$ so that $\mu = f\left(e^{\sqrt{-1}\theta}\right)d\theta/2\pi$ and $\delta \le f(\zeta) \le \delta^{-1}$ on $\mathbf{T}$ for some $\delta > 0$. For each $n \in \mathbf{N}$ choose

$$0 = b_0^{(n)} < a_1^{(n)} < b_1^{(n)} < a_2^{(n)} < b_2^{(n)} < \cdots < a_n^{(n)} < b_n^{(n)} = 2\pi$$

such that

$$\frac{1}{2\pi}\int_0^{a_j^{(n)}} f\Big(e^{\sqrt{-1}\theta}\Big)\,d\theta = \frac{j-\frac{1}{2}}{n}, \quad \frac{1}{2\pi}\int_0^{b_j^{(n)}} f\Big(e^{\sqrt{-1}\theta}\Big)\,d\theta = \frac{j}{n};$$

hence

$$\frac{\pi\delta}{n} \le b_j^{(n)} - a_j^{(n)} \le \frac{\pi}{n\delta}, \quad \frac{\pi\delta}{n} \le a_j^{(n)} - b_{j-1}^{(n)} \le \frac{\pi}{n\delta} \tag{A.1}$$

for all $1 \le j \le n$. Define

$$\begin{aligned}
\Delta_n &:= \Big\{\Big(e^{\sqrt{-1}\theta_1},\dots,e^{\sqrt{-1}\theta_{n-1}}\Big\}\Big) : a_j^{(n)} \le \theta_j \le b_j^{(n)},\ 1 \le j \le n-1\big\}, \\
\Theta_n &:= \big\{(\theta_1,\dots,\theta_{n-1}) : a_j^{(n)} \le \theta_j \le b_j^{(n)},\ 1 \le j \le n-1\big\}, \\
\xi_i^{(n)} &:= \max\Big\{Q\Big(e^{\sqrt{-1}\theta}\Big) : a_i^{(n)} \le \theta \le b_i^{(n)}\Big\} \ \text{ for } \ 1 \le i \le n-1, \\
d_{ij}^{(n)} &:= \min\Big\{\Big|e^{\sqrt{-1}s} - e^{\sqrt{-1}t}\Big| : a_i^{(n)} \le s \le b_i^{(n)},\ a_j^{(n)} \le t \le b_j^{(n)}\Big\} \ \text{ for } \ 1 \le i,j \le n-1.
\end{aligned}$$

For every neighborhood G of μ, if n is sufficiently large, then we have

$$\Delta_n \subset \left\{(\zeta_1,\dots,\zeta_{n-1}) \in \mathbf{T}^{n-1} : \frac{\delta_{\zeta_1}+\cdots+\delta_{\zeta_n}}{n} \in G\right\}$$

so that with $\theta_n = -(\theta_1+\cdots+\theta_{n-1})$

$$\begin{aligned}
\tilde{\lambda}_n^{\mathrm{SU}}(Q)&\left\{\frac{1}{n}\big(\delta_{\zeta_1}+\cdots+\delta_{\zeta_n}\big) \in G\right\} \ge \tilde{\lambda}_n^{\mathrm{SU}}(Q)(\Delta_n) \\
&= \frac{1}{\widetilde{Z}_n^{\mathrm{SU}}(Q)(2\pi)^{n-1}} \int\cdots\int_{\Theta_n} \exp\left(-n\sum_{i=1}^n Q\Big(e^{\sqrt{-1}\theta_i}\Big)\right) \\
&\qquad\qquad \times \prod_{1\le i<j\le n} \Big|e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j}\Big|^2\,d\theta_1\cdots d\theta_{n-1} \\
&\ge \frac{1}{\widetilde{Z}_n^{\mathrm{SU}}(Q)(2\pi)^{n-1}} \exp\left(-n\sum_{i=1}^{n-1}\xi_i^{(n)}\right) e^{-nM} \prod_{1\le i<j\le n-1} (d_{ij}^{(n)})^2 \\
&\qquad \times \int\cdots\int_{\Theta_n} \prod_{i=1}^{n-1} \Big|e^{\sqrt{-1}\theta_i} - e^{-\sqrt{-1}(\theta_1+\cdots+\theta_{n-1})}\Big|^2\,d\theta_1\cdots d\theta_{n-1},
\end{aligned}$$

where $M := \max\{Q(\zeta) : \zeta \in \mathbf{T}\}$. Notice

$$\Big\{\theta_1+\cdots+\theta_{n-1} : (\theta_1,\dots,\theta_{n-1}) \in \Theta_n\Big\} = \left[\sum_{i=1}^{n-1} a_i^{(n)}, \sum_{i=1}^{n-1} b_i^{(n)}\right],$$

and for n large enough

$$\sum_{i=1}^{n-1} b_i^{(n)} - \sum_{i=1}^{n-1} a_i^{(n)} \ge \frac{n-1}{n}\pi\delta > \frac{3\pi}{n\delta}. \tag{A.2}$$

From (A.1) and (A.2) we can choose an interval $[\alpha,\beta] \subset \Big[\sum_{i=1}^{n-1} a_i^{(n)}, \sum_{i=1}^{n-1} b_i^{(n)}\Big]$ such that $\beta - \alpha = \pi\delta/n^2$ and

$$[-\beta,-\alpha] \subset \left[b_{k-1}^{(n)} + \frac{\pi\delta}{n^2}, a_k^{(n)} - \frac{\pi\delta}{n^2}\right] \pmod{2\pi}$$

for some $1 \le k \le n$. Then, there exist subintervals $[\alpha_i, \beta_i] \subset \left[a_i^{(n)}, b_i^{(n)}\right]$, $1 \le i \le n-1$, such that

$$\beta_i - \alpha_i = \frac{\pi\delta}{n^2(n-1)}, \quad \sum_{i=1}^{n-1} \alpha_i = \alpha, \quad \sum_{i=1}^{n-1} \beta_i = \beta,$$

and hence

$$\begin{aligned}
&\int \cdots \int_{\Theta_n} \prod_{i=1}^{n-1} \left| e^{\sqrt{-1}\theta_i} - e^{-\sqrt{-1}(\theta_1 + \cdots + \theta_{n-1})} \right|^2 d\theta_1 \cdots d\theta_{n-1} \\
&\quad \ge \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_{n-1}}^{\beta_{n-1}} \left| e^{\sqrt{-1}\theta_i} - e^{-\sqrt{-1}(\theta_1 + \cdots + \theta_{n-1})} \right|^2 d\theta_1 \cdots d\theta_{n-1} \\
&\quad \ge \left(\frac{2\delta}{n^2}\right)^{2(n-1)} \left(\frac{\pi\delta}{n^2(n-1)}\right)^{n-1}.
\end{aligned}$$

Therefore, for sufficiently large n, we get

$$\begin{aligned}
&\tilde{\lambda}_n^{\mathrm{SU}}(Q)\left\{\frac{1}{n}\big(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n}\big) \in G\right\} \\
&\quad \ge \frac{(2\delta^3)^{n-1}}{\widetilde{Z}_n^{\mathrm{SU}}(Q) n^{7(n-1)}} \exp\left(-n \sum_{i=1}^{n-1} \xi_i^{(n)}\right) \prod_{1 \le i < j \le n-1} \left(d_{ij}^{(n)}\right)^2.
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{n\to\infty} \frac{2}{n^2} \sum_{1 \le i < j \le n-1} \log d_{ij}^{(n)} \\
&\quad = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f\left(e^{\sqrt{-1}s}\right) f\left(e^{\sqrt{-1}t}\right) \log \left| e^{\sqrt{-1}s} - e^{\sqrt{-1}t} \right| ds\, dt \\
&\quad = \iint_{\mathbf{T}^2} \log|\zeta - \eta|\, d\mu(\zeta)\, d\mu(\eta)
\end{aligned}$$

as well as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \xi_i^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} Q\left(e^{\sqrt{-1}s}\right) f\left(e^{\sqrt{-1}s}\right) ds = \int_{\mathbf{T}} Q(\zeta)\, d\mu(\zeta),$$

we have

$$\begin{aligned}
0 &\ge \limsup_{n\to\infty} \frac{1}{n^2} \log \tilde{\lambda}_n^{\mathrm{SU}}(Q)\left\{\frac{1}{n}\big(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n}\big) \in G\right\} \\
&\ge -\iint_{\mathbf{T}^2} F(\zeta, \eta)\, d\mu(\zeta)\, d\mu(\eta) - \liminf_{n\to\infty} \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q)
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n^2} \log \tilde{\lambda}_n^{\mathrm{SU}}(Q)\left\{\frac{1}{n}\big(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n}\big) \in G\right\} \\
&\quad \ge -\iint_{\mathbf{T}^2} F(\zeta, \eta)\, d\mu(\zeta)\, d\mu(\eta) - \limsup_{n\to\infty} \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q).
\end{aligned}$$

These imply (iii) and (iv). $\square$

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